PAPER

On the Capacity of MIMO Wireless Channels

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SUMMARY In this paper, we present a new closed-form formula for the ergodic capacity of multiple-input multiple-output (MIMO) wireless channels. Assuming independent and identically distributed (i.i.d.) Rayleigh flat-fading between antenna pairs and equal power allocation to each of the transmit antennas, the ergodic capacity of such channels is expressed in closed form as finite sums of the exponential integrals which are the special cases of the complementary incomplete gamma function. Using the *asymptotic capacity rate* of MIMO channels, which is defined as the asymptotic growth rate of the ergodic capacity, we also give simple approximate expressions for the MIMO capacity. Numerical results show that the approximations are quite accurate for the entire range of average signal-to-noise ratios.

key words: *channel capacity, multiple antennas, MIMO channels*

1. Introduction

In research areas on wireless communications, multiple-input multiple-output (MIMO) systems equipped with multiple antennas at both transmit and receive ends have recently drawn considerable attention in response to the increasing requirements on data rate and quality in radio links [1]–[9]. Multipath signal propagation has long been thought as an impairment limiting the system capacity and reliable communication in wireless channels, together with the constraint of power and bandwidth. In MIMO systems, this multipath propagation due to the rich scattering in wireless channels is used to improve achievable data rate and link quality.

Recent seminal work in [1] and [2] has shown that the use of multiple antennas at both the transmitter and the receiver significantly increases the information-theoretic capacity far beyond that of single-antenna systems. As the number of antennas at both the transmitter and the receiver gets larger, the capacity increases linearly with the minimum of the number of transmit and receive antennas for a given fixed signal-to-noise ratio (SNR) [1]–[3], even if the fades between antenna pairs are correlated [4]. In particular, Telatar [1] derived the analytic expression for the ergodic capacity of independent and identically distributed (i.i.d.) Rayleigh flat-fading MIMO channels in integral form involving the Laguerre polynomials and provided a look-up table obtained by numerically evaluating the integrals to find the ergodic capacity for different numbers of transmit and receive antennas. To date, and to the best knowledge of the authors, no closed-form expressions for the ergodic capacity of the MIMO channel are available for a finite number of antennas although those for the channel with multiple antennas at only one end of link have been reported in previous work [10]–[12].

The main objective of this paper is to extend the analysis in [1] to obtain closed-form expressions for the ergodic capacity of i.i.d. Rayleigh flat-fading MIMO channels. We also introduce the *capacity rate* which is defined as the growth rate of ergodic capacity with respect to the number of antennas. In random matrix theory (RMT) [16]–[19], it is well known that the eigenvalues of a large class of random matrix ensembles have fewer random fluctuations as the matrix dimension gets larger, i.e., the random distribution of the eigenvalues converges to a deterministic distribution in the limit of infinite matrix size. From these results of the RMT, previous studies [1], [4] have shown that the *asymptotic* capacity rate of MIMO channels converges to a nonzero constant determined by the average SNR and the limiting ratio between the numbers of transmit and receive antennas. Using this asymptotic capacity rate of MIMO channels, we present very accurate approximation formulas for the MIMO ergodic capacity. We also show that the capacity rates of single-input multiple-output (SIMO) and multiple-input single-output (SIMO) channels approach zero asymptotically as the number of antennas tends to infinity, in contrast to MIMO channels.

The remainder of this paper is organized as follows. The next section provides a brief review on the ergodic capacity for MIMO channels. In Sect. 3, we derive the closed-form formula for the MIMO ergodic capacity. In Sect. 4, we define the capacity rates of MIMO, MISO, and SIMO channels and analyze their asymptotic behavior as the number of antennas tends to infinity. In Sect. 5 we present approximate formulas for the MIMO ergodic capacity and conclusions are presented in Sect. 6.

2. Capacity of MIMO Channels

In this section, we briefly review the capacity formula for MIMO channels. Consider a point-to-point communication link with t transmit and r receive antennas. Throughout the paper we refer to $\alpha = \min\{t, r\}$ and $\beta = \max\{t, r\}$, and restrict our analysis to the frequency-flat fading case.

The total power of the complex transmitted signal vector $\mathbf{x} \in \mathbb{C}^t$ is constrained to P regardless of the number of antennas, namely

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$$E\left[\mathbf{x}^\dagger\mathbf{x}\right] \le P \tag{1}$$

where the superscript † denotes the transpose conjugate. At each symbol interval, the received signal vector $\mathbf{y} \in \mathbb{C}^r$ is given by

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \tag{2}$$

where $\mathbf{H} \in \mathbb{C}^{r\times t}$ is a $r \times t$ random channel matrix and $\mathbf{n}$ is a complex r dimensional additive white Gaussian noise (AWGN) vector with i.i.d. circularly symmetric Gaussian components, $E\left[\mathbf{n}\mathbf{n}^\dagger\right] = \sigma_n^2\mathbf{I}_r$†. The entries H_{ij}, $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, t$, of $\mathbf{H}$ are the complex channel gains between transmit antenna j and receive antenna i, which are modeled as i.i.d. complex Gaussian random variables with zero mean and unit variance, i.e., $H_{ij} \sim \mathcal{CN}(0, 1)$. In this case, the average SNR at each receive antenna is equal to $\rho = P/\sigma_n^2$. Furthermore, we assume that the channel is perfectly known to the receiver but unknown to the transmitter.

When the transmitted signal vector $\mathbf{x}$ is composed of t statistically independent equal power components each with a circularly symmetric complex Gaussian distribution, the channel capacity under transmit power constraint P is given by [1], [2]

$$C = \log_2 \det\left(\mathbf{I}_r + \frac{\rho}{t}\mathbf{H}\mathbf{H}^\dagger\right) \text{ bits/s/Hz}. \tag{3}$$

The ergodic (mean) capacity of the random MIMO channel, which is the Shannon capacity obtained by assuming it is possible to code over many independent channel realizations, is evaluated by averaging C with respect to the random matrix channel $\mathbf{H}$, namely [1]

$$\langle C\rangle_{t,r} = E\left[\log_2 \det\left(\mathbf{I}_r + \frac{\rho}{t}\mathbf{H}\mathbf{H}^\dagger\right)\right]. \tag{4}$$

For some specific models of the channel matrix, $\langle C\rangle_{t,r}$ can be evaluated by statistical simulations. However, these numerical matrix calculations may be very lengthy, especially when the number of antennas is very large.

Define

$$\mathbf{W} = \begin{cases} \mathbf{H}\mathbf{H}^\dagger & r < t \\ \mathbf{H}^\dagger\mathbf{H} & r \ge t, \end{cases} \tag{5}$$

then, the random matrix $\mathbf{W}$ has the central Wishart distribution with parameters α and β, and its random eigenvalues are of great interest in multivariate statistics (see [16]–[19], and references therein). Using the singular value decomposition theorem and the general results from the RMT, Telatar [1] derived the analytic formula for the ergodic capacity of MIMO channels in (4) as follows:

$$\langle C\rangle_{t,r} = \alpha \int_0^\infty \log_2(1 + \rho\lambda/t)\, p_\lambda(\lambda)\, d\lambda \tag{6}$$

where $p_\lambda(\lambda)$ is the distribution of a randomly selected eigenvalue of $\mathbf{W}$. The distribution $p_\lambda(\lambda)$ is given by [1]

$$p_\lambda(\lambda) = \frac{1}{\alpha}\sum_{k=0}^{\alpha-1} \frac{k!\lambda^{\beta-\alpha}e^{-\lambda}}{(\beta-\alpha+k)!}\left[L_k^{\beta-\alpha}(\lambda)\right]^2, \ \lambda \ge 0 \tag{7}$$

where $L_k^m(z)$ is the Laguerre polynomial of order k defined as [20, Eq. (8.970.1)]

$$\begin{aligned} L_k^m(z) &= \frac{1}{k!}e^z z^{-m}\frac{d^k}{dz^k}\left\{e^{-z}z^{k+m}\right\} \\ &= \sum_{i=0}^{k}(-1)^i\binom{k+m}{k-i}\frac{z^i}{i!}. \end{aligned} \tag{8}$$

where $\binom{n}{k} = n!/(k!(n-k)!)$ is the binomial coefficient.

3. Closed-Form Formulas for the Capacity

We now derive a closed-form expression for the capacity of MIMO channels.

Theorem 1: The ergodic capacity in bits/s/Hz of an i.i.d. Rayleigh fading MIMO channel with t transmit and r receive antennas under total transmit power constraint and equal power allocation is given by

$$\begin{aligned} \langle C\rangle_{t,r} = e^{t/\rho}\log_2(e)\sum_{k=0}^{\alpha-1}\sum_{l=0}^{k}\sum_{i=0}^{2l}\Bigg\{&\frac{(-1)^i(2l)!}{2^{2k-i}l!i!} \\ &\times\frac{(\beta-\alpha+i)!}{(\beta-\alpha+l)!}\binom{2k-2l}{k-l}\binom{2\beta-2\alpha+2l}{2l-i} \\ &\times\sum_{j=0}^{\beta-\alpha+i}E_{j+1}\left(\frac{t}{\rho}\right)\Bigg\} \end{aligned} \tag{9}$$

where $E_n(z)$ is the exponential integral of order n defined by [20, p.xxxv]

$$E_n(z) = \int_1^\infty e^{-zu}u^{-n}du, \quad n = 0, 1, \cdots, \mathrm{Re}[z] > 0. \tag{10}$$

Proof: From (8) and the identity of [20, Eq. (8.976.3)]

$$\left[L_k^m(z)\right]^2 = \frac{\Gamma(k+m+i)}{2^{2k}k!}\sum_{l=0}^{k}\frac{(2l)!\binom{2k-2l}{k-l}}{l!\Gamma(m+l+1)}L_{2l}^{2m}(2z) \tag{11}$$

where $\Gamma(z)$ is the gamma function, the distribution $p_\lambda(\lambda)$ can be rewritten as

$$\begin{aligned} p_\lambda(\lambda) &= \frac{1}{\alpha}\sum_{k=0}^{\alpha-1}\frac{k!\lambda^{\beta-\alpha}e^{-\lambda}}{(\beta-\alpha+k)!}\left[L_k^{\beta-\alpha}(\lambda)\right]^2 \\ &= \frac{1}{\alpha}\sum_{k=0}^{\alpha-1}\sum_{l=0}^{k}\frac{(2l)!\binom{2k-2l}{k-l}\lambda^{\beta-\alpha}e^{-\lambda}}{2^{2k}l!(\beta-\alpha+l)!}L_{2l}^{2\beta-2\alpha}(2\lambda) \\ &= \frac{1}{\alpha}\sum_{k=0}^{\alpha-1}\sum_{l=0}^{k}\sum_{i=0}^{2l}\Bigg\{\frac{(-1)^i(2l)!\lambda^{\beta-\alpha+i}e^{-\lambda}}{2^{2k-i}l!i!(\beta-\alpha+l)!} \\ &\quad\times\binom{2k-2l}{k-l}\binom{2\beta-2\alpha+2l}{2l-i}\Bigg\}, \ \lambda \ge 0. \end{aligned} \tag{12}$$

Substituting (12) into (6), the ergodic capacity is written as

†$\mathbf{I}_n$ represents the $n \times n$ identity matrix.

$$\langle C\rangle_{t,r} = \log_2(e)\sum_{k=0}^{\alpha-1}\sum_{l=0}^{k}\sum_{i=0}^{2l}\Bigg\{\frac{(-1)^i(2l)!}{2^{2k-i}l!i!(\beta-\alpha+l)!} \times\binom{2k-2l}{k-l}\binom{2\beta-2\alpha+2l}{2l-i} \times\underbrace{\int_0^\infty \ln(1+\rho\lambda/t)\lambda^{\beta-\alpha+i}e^{-\lambda}d\lambda}_{\triangleq I}\Bigg\}. \quad (13)$$

To evaluate the integral I in (13), we use the following result from [12, Appendix B]:

$$\mathcal{I}_n(\mu) = \int_0^\infty \ln(1+z)z^{n-1}e^{-\mu z}dz,\ \mu>0, n=1,2,\cdots$$
$$= (n-1)!e^\mu \sum_{j=1}^{n}\frac{\Gamma(-n+j,\mu)}{\mu^j} \quad (14)$$

where $\Gamma(a,z)$ is the complementary incomplete gamma function defined by [20, Eq. (8.350.2)]

$$\Gamma(a,z) = \int_z^\infty e^{-u}u^{a-1}du. \quad (15)$$

According to (14), the integral I is evaluated as

$$I = (\beta-\alpha+i)!\left(\frac{t}{\rho}\right)^{\beta-\alpha+i+1} e^{t/\rho} \times\sum_{j=1}^{\beta-\alpha+i+1}\frac{\Gamma(-\beta+\alpha-i-1+j,t/\rho)}{(t/\rho)^j}. \quad (16)$$

The exponential integral $E_n(z)$ of (10) is the special case of the complementary incomplete gamma function, i.e.,

$$E_n(z) = z^{n-1}\Gamma(1-n,z). \quad (17)$$

Applying the identity (17) to (16), we have

$$I = (\beta-\alpha+i)!e^{t/\rho}\sum_{j=0}^{\beta-\alpha+i}E_{j+1}\left(\frac{t}{\rho}\right). \quad (18)$$

Inserting (18) into (13), the ergodic capacity $\langle C\rangle_{t,r}$ in bits/s/Hz can be expressed in closed form as (9). □

Example 1: Consider $t = r = n$. From (9) with $\alpha=\beta=n$, the capacity of a MIMO channel with n antennas at both the transmitter and the receiver is given by

$$\langle C\rangle_{n,n} = e^{n/\rho}\log_2(e)\sum_{k=0}^{n-1}\sum_{l=0}^{k}\sum_{i=0}^{2l}\Bigg\{\frac{(-1)^i}{2^{2k-i}}\binom{2l}{l} \times\binom{2k-2l}{k-l}\binom{2l}{i}\sum_{j=0}^{i}E_{j+1}\left(\frac{n}{\rho}\right)\Bigg\}. \quad (19)$$

Example 2: Consider $t = n$ and $r = 1$ (MISO channel). From (9) with $\alpha = 1$ and $\beta = n$, the capacity of a MISO channel with n transmit antennas is given by

$$\langle C\rangle_{n,1} = e^{n/\rho}\log_2(e)\sum_{j=0}^{n-1}E_{j+1}\left(\frac{n}{\rho}\right). \quad (20)$$

Example 3: Consider $t = 1$ and $r = n$ (SIMO channel). From (9) with $\alpha = 1$ and $\beta = n$, the capacity of a SIMO channel with n receive antennas is given by

$$\langle C\rangle_{1,n} = e^{1/\rho}\log_2(e)\sum_{j=0}^{n-1}E_{j+1}\left(\frac{1}{\rho}\right). \quad (21)$$

Note that increasing the number of receive antennas from $n-1$ to n in the SIMO channel yields an additional capacity advantage equal to $e^{1/\rho}\log_2(e)E_n(1/\rho)$ bits/s/Hz and (21) is in agreement with the formerly known result of the exact capacity formula for Rayleigh fading channels with reception diversity if applying the identity (17) to [12, Eq. (40)].

4. Capacity Rates

A natural question to ask is "How does the capacity grow with the number of antennas." We define the capacity rate as this rate of growth as follows.

Definition 1: For a MIMO channel with t transmit and r receive antennas, the capacity rate, denoted by $\mathcal{R}_{\text{MIMO}}$, is defined as its ergodic capacity normalized by $\min\{t,r\}$. The *asymptotic* capacity rate for the MIMO channel is defined as the asymptotic behavior of $\mathcal{R}_{\text{MIMO}}$, i.e.,

$$\mathcal{R}^*_{\text{MIMO}}(\tau) \triangleq \lim_{\substack{t,r\to\infty\\ r/t\to\tau}}\frac{\langle C\rangle_{t,r}}{\min\{t,r\}} \quad (22)$$

which implies the asymptotic growth rate of the MIMO ergodic capacity when t and r tend to infinity in such a way r/t tends to a limit τ. For MISO and SIMO channels, the capacity rates, denoted by $\mathcal{R}_{\text{MISO}}$ and $\mathcal{R}_{\text{SIMO}}$, are defined as their ergodic capacity normalized by the number of receive or transmit antennas, respectively. The *asymptotic* capacity rates for the MISO and SIMO channels are defined as the asymptotic behavior of $\mathcal{R}_{\text{MISO}}$ and $\mathcal{R}_{\text{SIMO}}$, i.e.,

$$\mathcal{R}^*_{\text{MISO}} \triangleq \lim_{t\to\infty}\frac{\langle C\rangle_{t,1}}{t} \quad (23)$$

and

$$\mathcal{R}^*_{\text{SIMO}} \triangleq \lim_{r\to\infty}\frac{\langle C\rangle_{1,r}}{r}. \quad (24)$$

Using the limit theorem on the distribution of the eigenvalues of large dimensional random matrices [19], the asymptotic behavior of the MIMO capacity has been reported in [1] and [4]. The following theorem determines the asymptotic capacity rate for MIMO channels and is used to establish approximate formulas for (9) in the next section.

Theorem 2: If t and r tend to infinity in such a way that $r/t \to \tau$ and let $\tilde{\tau} = \tau^{\text{sgn}(\tau-1)}$, $a(\tau) = (\sqrt{\tilde{\tau}}-1)^2$, $b(\tau) = (\sqrt{\tilde{\tau}}+1)^2$, and

$$\nu = \begin{cases} \rho\tau & \tau<1 \\ \rho & \tau\geq 1, \end{cases}$$

then the asymptotic capacity rate $\mathcal{R}^*_{\text{MIMO}}(\tau)$ is given by

$$\begin{aligned}\mathcal{R}^*_{\text{MIMO}}(\tau) &= \frac{1}{2\pi}\int_{a(\tau)}^{b(\tau)} \log_2(1+\nu z) \\ &\quad \times \sqrt{\left(\frac{b(\tau)}{z}-1\right)\left(1-\frac{a(\tau)}{z}\right)}\, dz \qquad (25)\\ &= \tilde{\tau}\log_2\left\{1+\nu-\mathcal{J}(\nu,\tilde{\tau})\right\} \\ &\quad +\log_2\left\{1+\nu\tilde{\tau}-\mathcal{J}(\nu,\tilde{\tau})\right\} \\ &\quad -\frac{\log_2(e)}{\nu}\mathcal{J}(\nu,\tilde{\tau}) \qquad (26)\end{aligned}$$

where

$$\mathcal{J}(u,w) \triangleq \frac{1}{4}\left\{\sqrt{u\left(\sqrt{w}+1\right)^2+1} - \sqrt{u\left(\sqrt{w}-1\right)^2+1}\right\}^2. \qquad (27)$$

Proof: The well-known result from the RMT [19] says that as $\beta/\alpha \to \tilde{\tau}$ (or $r/t \to \tau$), the asymptotic eigenvalue density of $\frac{1}{\alpha}\mathbf{W}$ converges in probability to

$$p^*(z) = \frac{1}{2\pi}\sqrt{\left(\frac{b(\tau)}{z}-1\right)\left(1-\frac{a(\tau)}{z}\right)}, \qquad a(\tau) \le z \le b(\tau)$$

which is sometimes called the deformed quarter circle law or Marcenko-Pastur law. From (6) and $p^*(z)$, the asymptotic capacity rate defined in (22) can be written as

$$\begin{aligned}\mathcal{R}^*_{\text{MIMO}}(\tau) &= \frac{1}{2\pi}\int_{a(\tau)}^{b(\tau)} \log_2\left(1+\frac{\rho\alpha}{t}z\right)p^*(z)\,dz \\ &= \frac{1}{2\pi}\int_{a(\tau)}^{b(\tau)} \log_2(1+\nu z)\,p^*(z)\,dz.\end{aligned}$$

□

Note that the closed-form expression (26) for the integral of (25) was presented in the context of the capacity analysis for code-division multiple-access (CDMA) systems with random spreading sequences [13], [14]. In [13], (26) was indirectly obtained by using the fact that the perfect successive canceler with minimum mean-square-error (MMSE) prefiltering achieves asymptotically the same capacity as the maximum-likelihood decoder. On the order hand, Rapajic and Popescu [14] derived the closed-form expression directly from the integral and proved the results in [13]. When $t = r$, i.e., $\tau = 1$, (26) reduces to[†]

$$\begin{aligned}\mathcal{R}^*_{\text{MIMO}}(1) &= \lim_{r\to\infty}\frac{\langle C\rangle_{r,r}}{r} \\ &= 2\log_2\left(\sqrt{4\rho+1}+1\right) \\ &\quad -\frac{\log_2(e)}{4\rho}\left(\sqrt{4\rho+1}-1\right)^2-2. \qquad (28)\end{aligned}$$

Theorem 3: For MISO and SIMO channels, the asymptotic capacity rate is equal to zero, that is, $\mathcal{R}^*_{\text{MISO}} = \mathcal{R}^*_{\text{SIMO}} = 0$.

Proof: As t gets large, $\frac{1}{t}\mathbf{H}\mathbf{H}^\dagger$ for fixed r converges almost surely to $\mathbf{I}_r$ by the law of large numbers. Therefore, the ergodic capacity for large t and fixed r becomes

$$\begin{aligned}\langle C\rangle_{t,r} &\to E\left[\log_2\det\left(\mathbf{I}_r+(1+\rho)\right)\right] \\ &= r\cdot\log_2(1+\rho). \qquad (29)\end{aligned}$$

Similarly, as r gets large, $\frac{1}{r}\mathbf{H}^\dagger\mathbf{H}$ for fixed t converges almost surely to $\mathbf{I}_t$ by the law of large numbers, and the ergodic capacity for fixed t and large r becomes

$$\begin{aligned}\langle C\rangle_{t,r} &\to E\left[\log_2\det\left(\mathbf{I}_t+\left(1+\frac{\rho r}{t}\right)\right)\right] \\ &= t\cdot\log_2\left(1+\frac{\rho r}{t}\right). \qquad (30)\end{aligned}$$

From (23), (24), (29) and (30), it is straightforward that $\mathcal{R}^*_{\text{MISO}} = \mathcal{R}^*_{\text{SIMO}} = 0$. □

Figure 1 shows the capacity rates for MIMO, MISO, and SIMO channels versus the number of antennas n at ρ =10 dB for the following five cases: (a) $t = n$ and $r = 2n$; (b) $t = 2n$ and $r = n$; (c) $t = r = n$; (d) $t = 1$ and $r = n$ (SIMO); (e) $t = n$ and $r = 1$ (MISO). If the number of antennas is increased at both the transmitter and the receiver, i.e., the cases (a), (b), and (c), the MIMO capacity rate $\mathcal{R}_{\text{MIMO}}$ converges very fast to its asymptotic value according to Theorem 2 as n increases. The asymptotic capacity rates for (a), (b), and (c) are 4.0114, 3.1253, and 2.7233 bits/s/Hz, respectively. On the order hand, for SIMO and MISO channels with n receive or transmit antennas, i.e., the cases (d) and (e), respectively, the capacity rate $\mathcal{R}_{\text{SIMO}}$ and $\mathcal{R}_{\text{MISO}}$ decrease with n and approach zero as $n \to \infty$. This implies

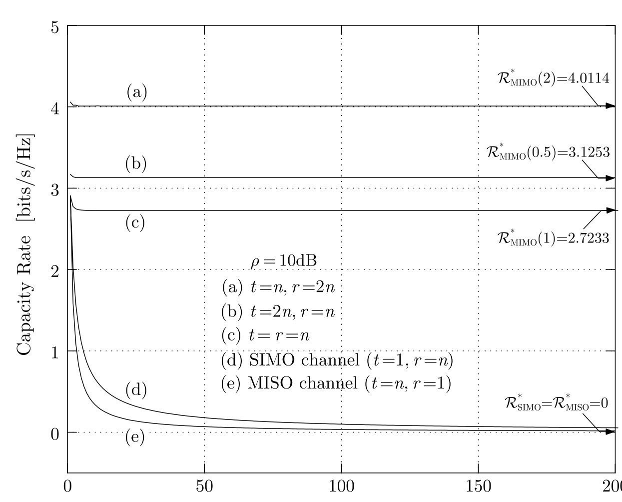

Fig. 1 Capacity rates for MIMO, MISO, and SIMO channels versus the number of antennas n at ρ =10 dB.

[†]There exists a minor typo in [4, Eq. (9)] (the term −2 in (28) was missed out).

that if the number of antennas is increased only at the transmitter or the receiver, the asymptotic capacity rate is equal to zero as shown in Theorem 3. Also, we see that due to the total transmitted power constraint, $\mathcal{R}_{\text{MISO}}$ converges to zero faster than $\mathcal{R}_{\text{SIMO}}$ as the number of antennas increases.

5. Approximate Formulas for the Capacity

From the fact that the capacity rate for MIMO channels with a certain fixed ratio τ converges very quickly to its asymptotic value $\mathcal{R}^*_{\text{MIMO}}(\tau)$ as the number of antennas increases (see Fig. 1), it is obvious that the capacity of MIMO channels is very well approximated by a linear function of $\min\{t, r\}$ with a rate of $\mathcal{R}^*_{\text{MIMO}}(\tau)$ for a given average SNR. For example, if $t = r = n$, the capacity approximately increases $\mathcal{R}^*_{\text{MIMO}}(1)$ bits/s/Hz for each increase in n and thus $\langle C\rangle_{n,n} \approx \langle C\rangle_{1,1} + (n-1)\cdot\mathcal{R}^*_{\text{MIMO}}(1)$. Therefore, the following approximations are straightforward.

The ergodic capacity of a MIMO channel with n antennas at both the transmitter and the receiver in (19) is simply approximated by

$$\begin{aligned}\langle C\rangle_{n,n} &\approx \langle C\rangle_{1,1} + (n-1)\cdot\mathcal{R}^*_{\text{MIMO}}(1)\\ &= e^{1/\rho}\log_2(e)E_1\left(\frac{1}{\rho}\right)\\ &\quad +(n-1)\Bigg\{2\log_2\left(\sqrt{4\rho+1}+1\right)\\ &\quad -\frac{\log_2(e)}{4\rho}\left(\sqrt{4\rho+1}-1\right)^2-2\Bigg\}. \end{aligned} \tag{31}$$

For $t < r$, an approximate formula for the ergodic capacity is given by

$$\begin{aligned}\langle C\rangle_{t,r} &\approx \langle C\rangle_{1,\lceil r/t\rceil} + (t-1)\cdot\mathcal{R}^*_{\text{MIMO}}\left(\frac{r}{t}\right)\\ &= e^{1/\rho}\log_2(e)\sum_{j=0}^{\lceil r/t\rceil-1}E_{j+1}\left(\frac{1}{\rho}\right)\\ &\quad +(t-1)\Bigg[\frac{r}{t}\cdot\log_2\left\{1+\rho-\mathcal{J}\left(\rho,\frac{r}{t}\right)\right\}\\ &\quad +\log_2\left\{1+\frac{\rho r}{t}-\mathcal{J}\left(\rho,\frac{r}{t}\right)\right\}\\ &\quad -\frac{\log_2(e)}{\rho}\mathcal{J}\left(\rho,\frac{r}{t}\right)\Bigg] \end{aligned} \tag{32}$$

where $\lceil z\rceil$ represents the smallest integer greater than or equal to z. Similarly, when $t > r$, the ergodic capacity is approximated by

$$\begin{aligned}\langle C\rangle_{t,r} &\approx \langle C\rangle_{\lceil t/r\rceil,1} + (r-1)\cdot\mathcal{R}^*_{\text{MIMO}}\left(\frac{r}{t}\right)\\ &= e^{\lceil t/r\rceil/\rho}\log_2(e)\sum_{j=0}^{\lceil t/r\rceil-1}E_{j+1}\left(\frac{\lceil t/r\rceil}{\rho}\right)\\ &\quad +(r-1)\Bigg[\frac{t}{r}\cdot\log_2\left\{1+\frac{\rho r}{t}-\mathcal{J}\left(\frac{\rho r}{t},\frac{t}{r}\right)\right\}\\ &\quad +\log_2\left\{1+\rho-\mathcal{J}\left(\frac{\rho r}{t},\frac{t}{r}\right)\right\}\\ &\quad -\frac{t}{r}\frac{\log_2(e)}{\rho}\mathcal{J}\left(\frac{\rho r}{t},\frac{t}{r}\right)\Bigg]. \end{aligned} \tag{33}$$

To assess the accuracy of the approximate expressions for the ergodic capacity of MIMO channels, we compare the approximations with the exact calculations in Figs. 2–4. Figure 2 shows the exact ergodic capacity of a MIMO channel with $t = r = n$ and its approximation from (31) versus average SNR for various n. Figure 3 shows the exact ergodic capacity of a MIMO channel with $t = n$ and $r = \kappa\cdot n$ and its approximation from (32) versus average SNR for $n = 2, 6, 10$, and $\kappa = 1.5$ and 3. Figure 4 shows the exact

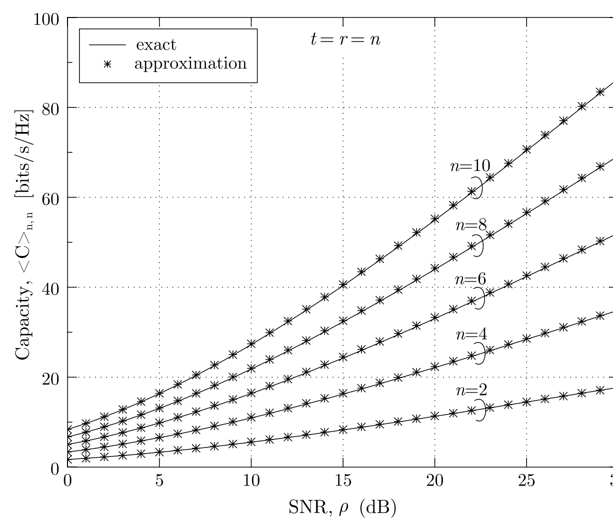

Fig. 2 Ergodic capacity and its approximation for a MIMO channel with $t = r = n$ versus average SNR ρ.

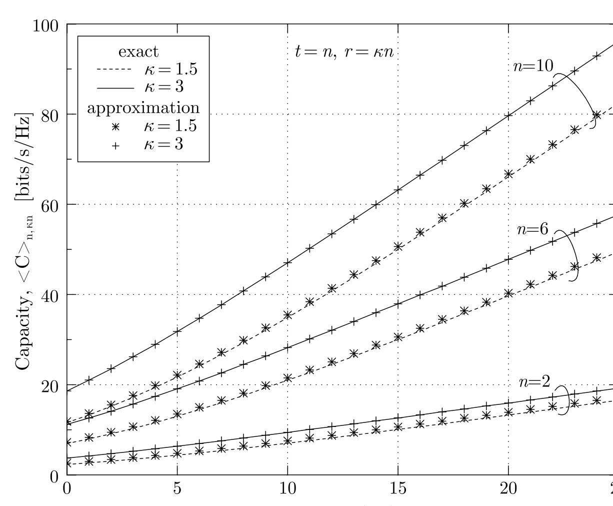

Fig. 3 Ergodic capacity and its approximation for a MIMO channel with $t = n$ and $r = \kappa\cdot n$ versus average SNR ρ.

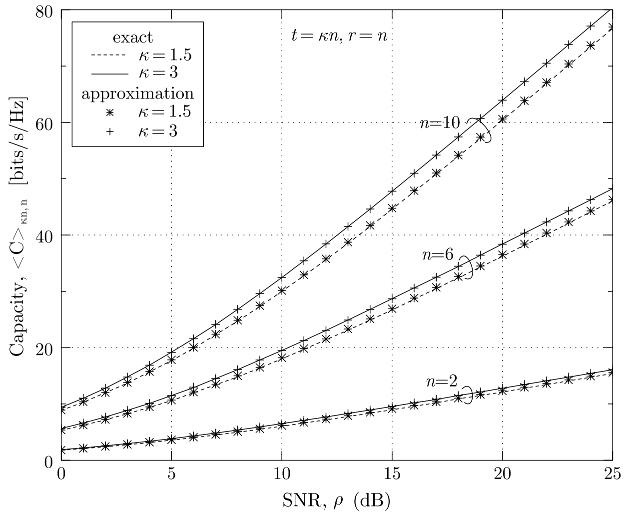

Fig. 4 Ergodic capacity and its approximation for a MIMO channel with $t = \kappa \cdot n$ and $r = n$ versus average SNR ρ.

ergodic capacity for $t = \kappa \cdot n$ and $r = n$ and its approximation from (33) versus average SNR for the same values of n and κ as those in Fig. 3. From these plots we can see that the approximations closely match the exact ergodic capacity (9) for the entire range of average SNRs and get more accurate if the number of transmit antennas is an integral multiple of that of receive antennas and vice versa. These closed-form approximations can therefore be safely used to predict the capacity of MIMO channels.

6. Conclusions

In this paper we obtained a closed-form expression and an accurate approximation for the ergodic capacity of i.i.d. Rayleigh flat-fading MIMO channels under total transmit power constraint and equal power allocation. By using these expressions, we can easily predict the capacity performance of MIMO channels without any numerical integrations or statistical simulations with numerical matrix calculations. We also defined the capacity rate as the growth rate of the ergodic capacity with respect to the number of antennas. The capacity rate converges asymptotically to a certain nonzero constant as the number of antennas at both the transmitter and receiver tends to infinity, while it approaches zero if the number of antennas tends to infinity only at transmitter or the receiver.

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References

[1] I.E. Telatar, "Capacity of multi-antenna Gaussian channels," European Trans. Telecommun. (ETT), vol.10, no.6, pp.586–595, Nov./Dec. 1999, originally published as Tech. Memo., Bell Laboratories, Lucent Technologies, Oct. 1995.

[2] G.J. Foschini and M.J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," Wireless Personal Commun., vol.6, no.3, pp.311–335, March 1998.

[3] G.G. Raleigh and J.M. Cioffi, "Spatio-temporal coding for wireless communications," IEEE Trans. Commun., vol.46, no.3, pp.357–366, March 1998.

[4] C.-N. Chuah, D. Tse, J.M. Kahn, and R.A. Valenzuela, "Capacity scaling in MIMO wireless systems under correlated fading," IEEE Trans. Inf. Theory, vol.48, no.3, pp.637–650, March 2002.

[5] G.J. Foschini, "Layered space-time architecture for wireless communication in a fading environment when using multi-element antennas," Bell Labs. Tech. J., vol.1, no.2, pp.41–59, 1996.

[6] V. Tarokh, N. Seshadri, and A.R. Calderbank, "Space-time codes for high data rate wireless communication: Performance criterion and code construction," IEEE Trans. Inf. Theory, vol.44, no.2, pp.744–765, March 1998.

[7] J.-C. Guey, M.P. Fitz, M.R. Bell, and W.-Y. Kuo, "Signal design for transmitter diversity wireless communication systems," IEEE Trans. Commun., vol.47, no.4, pp.527–537, April 1999.

[8] S.M. Alamouti, "A simple transmit diversity technique for wireless communications," IEEE J. Sel. Areas Commun., vol.16, no.8, pp.1451–1458, Oct. 1998.

[9] V. Tarokh, H. Jafarkhani, and A.R. Calderbank, "Space-time block codes from orthogonal designs," IEEE Trans. Inf. Theory, vol.45, no.5, pp.1456–1467, July 1999.

[10] W.C.Y. Lee, "Estimate of channel capacity in Rayleigh fading environment," IEEE Trans. Veh. Technol., vol.39, no.3, pp.187–189, Aug. 1990.

[11] C.G. Gunther, "Comment on estimate of channel capacity in Rayleigh fading environment," IEEE Trans. Veh. Technol., vol.45, no.2, pp.401–403, May 1996.

[12] M.-S. Alouini and A.J. Goldsmith, "Capacity of Rayleigh fading channels under different adaptive transmission and diversity-combining techniques," IEEE Veh. Technol., vol.48, no.4, pp.1165–1181, July 1999.

[13] S. Verdu and S. Shamai, "Spectral efficiency of CDMA with random spreading," IEEE Trans. Inf. Theory, vol.45, no.2, pp.622–640, March 1999.

[14] P.B. Rapajic and D. Popescu, "Information capacity of a random signature multiple-input multiple-output channel," IEEE Trans. Commun., vol.48, no.8, pp.1245–1248, Aug. 2000.

[15] T.M. Cover and J.A. Thomas, Elements of Information Theory, Wiley, New York, 1991.

[16] R.J. Muirhead, Aspects of Multivariate Statistical Theory, John Wiley & Sons, New York, 1982.

[17] V.L. Girko, Theory of Random Determinants, Kluwer, Norwell, MA, 1990.

[18] A. Edelman, Eigenvalues and Condition Numbers of Random Matrices, Ph.D. Thesis, Dept. Mathematics, M.I.T., Cambridge, MA, May 1989.

[19] J.W. Silverstein, "Strong convergence of the empirical distribution of eigenvalues of large dimensional random matrices," J. Multivariate Anal., vol.55, pp.331–339, 1995.

[20] I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series, and Products, 6th ed., Academic, San Diego, CA, 2000.

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