On the Error Probability of Binary and M-ary Signals in Nakagami-m Fading Channels

Hyundong Shin, *Student Member, IEEE,* and Jae Hong Lee, *Senior Member, IEEE*

***Abstract*—In this letter, we present new closed-form formulas for the exact average symbol-error rate (SER) of binary and M-ary signals over Nakagami-m fading channels with *arbitrary* fading index m. Using the well-known moment generating function-based analysis approach, we express the average SER in terms of the higher transcendental functions such as the Gauss hypergeometric function, Appell hypergeometric function, or Lauricella function. The results are generally applicable to arbitrary real-valued m. Furthermore, with the aid of reduction formulas of hypergeometric functions, we show previously published results for Rayleigh fading ($m = 1$) as special cases of our expressions.**

***Index Terms*—Digital communications, Nakagami fading, symbol-error rate (SER), wireless communications.**

I. Introduction

IN DIGITAL communication systems, the symbol-error rate (SER) has been used very extensively as a performance measure, and accurate methods for evaluating it over fading channels has been an area of long-time interest (see [2]–[9] and references therein). Recently, a unified approach for evaluating the error performance over fading channels has been developed by using alternative representations of the Gaussian and Marcum Q-functions [7]–[9]. By their alternative representations, the resulting expressions for average error rates are in the form of single finite-range integrals, whose integrand contains the moment generating function (MGF) of the instantaneous signal-to-noise ratio (SNR). In particular, Annamalai and Tellambura [7] derived closed-form solutions to the average SER for a broad class of binary and M-ary modulation formats in Nakagami-m fading with positive integer m, using some trigonometric identities and the MGF-based analysis method. They also extended the results to multichannel diversity reception. However, to the best knowledge of the authors, no closed-form SER expressions for M-ary signals are available for arbitrary M, except for noncoherent detection of orthogonal M-ary frequency-shift keying (MFSK), when the Nakagami fading index m is not restricted to positive integer values.

In this letter, using the MGF method and transforming single integrals into the hypergeometric functions [10], [11], we derive the exact and closed-form expressions for average SER of binary and M-ary signals in Nakagami-m fading channels with *arbitrary real-valued* m. Our approach leads to expressions of average SER involving the Gauss hypergeometric function ${}_2F_1$, Appell hypergeometric function F_1, or Lauricella function $F_D^{(n)}$.[1] In addition, we present some reduction formulas for F_1 and $F_D^{(n)}$. Using these reduction formulas and known identities and transformations of the hypergeometric functions, we show the well-known results for Rayleigh fading ($m = 1$) are special cases of our expressions.

II. Channel Model

Assume that the transmitted signal is received over slowly varying flat-fading channels, and let γ denote the instantaneous SNR defined by $\gamma \triangleq \alpha^2 E_s/N_0$ where α is the fading amplitude, E_s is the energy per symbol, and N_0 is the one-sided noise spectral density. For Nakagami-m fading, the probability density function (pdf) of α is given by [1]

$$p_\alpha(\alpha) = \frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m \alpha^{2m-1} e^{-m\alpha^2/\Omega}, \quad \alpha \geq 0 \tag{1}$$

where $\Gamma(\cdot)$ is the gamma function, $\Omega \triangleq E\left[\alpha^2\right]$ denotes the mean square value, and m is the fading severity parameter that ranges from 0.5 to ∞. Then, the pdf and MGF of γ are given by, respectively [7], [8]

$$p_\gamma(\gamma) = \frac{1}{\Gamma(m)}\left(\frac{m}{\overline{\gamma}}\right)^m \gamma^{m-1} e^{-m\gamma/\overline{\gamma}}, \quad \gamma \geq 0 \tag{2}$$

$$\phi_\gamma(s) \triangleq \int_0^\infty e^{-s\gamma} p_\gamma(\gamma)\, d\gamma = \left(1 + \frac{s\overline{\gamma}}{m}\right)^{-m}, \quad m \geq \frac{1}{2} \tag{3}$$

where $\overline{\gamma} \triangleq \Omega E_s/N_0$ denotes the average SNR per symbol.

III. Average SERs

In this section, we derive closed-form error rates for several modulation/detection schemes in Nakagami-m fading with arbitrary real-valued m, using the MGF-based analysis method and the transformation of single integrals into the hypergeometric functions.

A. Coherent BPSK and BFSK

The average bit-error rate (BER) for coherent binary signals is given by [7]

$$P_b(E) = \frac{1}{\pi}\int_0^{\pi/2} \phi_\gamma\left(\frac{g}{\sin^2\theta}\right) d\theta \tag{4}$$

where $g = 1$ for coherent binary phase-shift keying (BPSK), $g = 1/2$ for coherent orthogonal binary frequency-shift keying

[1]The hypergeometric functions are provided as the library functions in a common mathematical software package such as MATHEMATICA.

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The authors are with the School of Electrical Engineering, Seoul National University, Seoul 151-742, Korea (e-mail: shd71@snu.ac.kr).

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(BFSK), and $g = 0.715$ for coherent BFSK with minimum correlation. By change of the variable $t = \cos^2\theta$, after some manipulations, (4) can be expressed in closed form as

$$P_b(E) = \frac{\phi_\gamma(g)}{2\pi}\int_0^1 t^{-1/2}(1-t)^{m-1/2}\left(1-\frac{t}{1+\frac{g\overline{\gamma}}{m}}\right)^{-m} dt = \frac{\phi_\gamma(g)\,\Gamma\left(m+\frac{1}{2}\right)}{2\sqrt{\pi}\,\Gamma(m+1)}\,{}_2F_1\left(m,\frac{1}{2};m+1;\frac{1}{1+\frac{g\overline{\gamma}}{m}}\right) \quad (5)$$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometic function [10, eq. 2.12.(1)]. Note that we can easily show that (5) is equivalent to [7, eq. (16)] using the linear transformation ${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z)$ [10, eq. 2.1.(23)]. Making use of the identity ${}_2F_1(1,1/2;2;z) = 2\left(1-\sqrt{1-z}\right)/z$, we see that (5) reduces to the familiar results [2, eqs. (14-3-7) and (14-3-8)] for Rayleigh fading.

B. Noncoherent Detection of Equiprobable Correlated Binary Signals and $\pi/4$-Differential Quaternary Phase-Shift Keying (DQPSK)

Nonorthogonal signals can occur in either one of two cases; the signals can be chosen to be nonorthogonal at the transmitter or the orthogonal signals are transmitted, but due to imperfections in the receiver such as frequency or timing error, the signals become correlated, and thus, nonorthogonal.

The average BER for equal energy, equiprobable, correlated binary signals with noncoherent detection is given by [7]

$$P_b(E) = \frac{1}{2\pi}\int_0^\pi \phi_\gamma\left(\frac{\left(b^2-a^2\right)^2}{2\left(a^2+b^2\right)-4ab\cos\theta}\right)d\theta \quad (6)$$

where

$$a = \sqrt{\frac{(1-\sqrt{1-|\rho|^2})}{2}}, \quad b = \sqrt{\frac{(1+\sqrt{1-|\rho|^2})}{2}}$$

and $0 \le |\rho| \le 1$ is the magnitude of the crosscorrelation coefficient between the two signals. By change of the variable $t = \cos^2\theta$, after some manipulations, (6) can be expressed in closed form as

$$P_b(E) = \frac{\phi_\gamma(g_{\mathrm{NCB}})}{2\pi} \times\int_0^1 t^{-1/2}(1-t)^{-1/2}\cdot\left(1-\frac{\frac{4ab}{(a+b)^2}}{1+\frac{g_{\mathrm{NCB}}\overline{\gamma}}{m}}t\right)^{-m}\left(1-\frac{4ab}{(a+b)^2}t\right)^{m}dt = \frac{1}{2}\phi_\gamma(g_{\mathrm{NCB}})F_1\left(\frac{1}{2},m,-m;1;\frac{\frac{4ab}{(a+b)^2}}{1+\frac{g_{\mathrm{NCB}}\overline{\gamma}}{m}},\frac{4ab}{(a+b)^2}\right) \quad (7)$$

where $g_{\mathrm{NCB}} = (b-a)^2/2$ and $F_1(a,b,b';c;x,y)$ is the Appell hypergeometric function [10, eq. 5.8.(5)]. The special case $\rho = 0$ corresponds to orthogonal BFSK with noncoherent detection, and using $F_1(a,b,b';c;0,0) = 1$, we see that (7) reduces in this particular case to the familiar expression [5, eq. (11)].

Furthermore, $a = \sqrt{2-\sqrt{2}}$ and $b = \sqrt{2+\sqrt{2}}$ correspond to $\pi/4$-DQPSK with Gray coding [7], [8]. In this case, by substituting $a = \sqrt{2-\sqrt{2}}$ and $b = \sqrt{2+\sqrt{2}}$ into (7), the average SER for $\pi/4$-DQPSK with Gray coding can be obtained as

$$P_s(E) = \frac{1}{2}\phi_\gamma(g_{\mathrm{DQPSK}})F_1\left(\frac{1}{2},m,-m;1;\frac{\frac{2\sqrt{2}}{(2+\sqrt{2})}}{1+\frac{g_{\mathrm{DQPSK}}\overline{\gamma}}{m}},\frac{2\sqrt{2}}{2+\sqrt{2}}\right) \quad (8)$$

where $g_{\mathrm{DQPSK}} = 2-\sqrt{2}$. For Rayleigh fading, we can reduce (8) to [8, eq. (8.176)] using the identity (19).

C. M-ary Phase-Shift Keying (MPSK)

The average SER for coherent MPSK signals is given by [7]

$$P_s(E) = \frac{1}{\pi}\int_0^{\pi-(\pi/M)}\phi_\gamma\left(\frac{g_{\mathrm{MPSK}}}{\sin^2\theta}\right)d\theta = \underbrace{\frac{1}{\pi}\int_0^{\pi/2}\phi_\gamma\left(\frac{g_{\mathrm{MPSK}}}{\sin^2\theta}\right)d\theta}_{\triangleq\mathcal{I}_{1,\mathrm{MPSK}}}+\underbrace{\frac{1}{\pi}\int_{\pi/2}^{\pi-(\pi/M)}\phi_\gamma\left(\frac{g_{\mathrm{MPSK}}}{\sin^2\theta}\right)d\theta}_{\triangleq\mathcal{I}_{2,\mathrm{MPSK}}} \quad (9)$$

where $g_{\mathrm{MPSK}} = \sin^2(\pi/M)$. Considering the similarity of $\mathcal{I}_{1,\mathrm{MPSK}}$ to (4) and making the change of variable $t = \cos^2\theta/\cos^2(\pi/M)$ in $\mathcal{I}_{2,\mathrm{MPSK}}$, after some manipulations, we obtain

$$P_s(E) = \frac{\Gamma\left(m+\frac{1}{2}\right)}{2\sqrt{\pi}\,\Gamma(m+1)}\phi_\gamma(g_{\mathrm{MPSK}}) \cdot {}_2F_1\left(m,\frac{1}{2};m+1;\frac{1}{1+\frac{g_{\mathrm{MPSK}}\overline{\gamma}}{m}}\right) + \frac{1}{\pi}\cos\left(\frac{\pi}{M}\right)\phi_\gamma(g_{\mathrm{MPSK}}) \cdot F_1\left(\frac{1}{2},m,\frac{1}{2}-m;\frac{3}{2};\frac{\cos^2\left(\frac{\pi}{M}\right)}{1+\frac{g_{\mathrm{MPSK}}\overline{\gamma}}{m}},\cos^2\left(\frac{\pi}{M}\right)\right). \quad (10)$$

Note that using the identity (20) along with

$$\frac{1}{\pi}\tan^{-1}\left\{\cot\left(\frac{\pi}{x}\right)\right\}+\frac{1}{2} = \frac{x-1}{x}, \quad x > 1$$

we see that (10) reduces to [8, eq. (8.112)] for Rayleigh fading.

D. M-ary Quadrature Amplitude Modulation (MQAM)

The average SER for coherent square MQAM signals is given by [7]

$$P_s(E) = \underbrace{\frac{4q}{\pi}\int_0^{\pi/2}\phi_\gamma\left(\frac{g_{\mathrm{MQAM}}}{\sin^2\theta}\right)d\theta}_{\triangleq\mathcal{I}_{1,\mathrm{MQAM}}}-\underbrace{\frac{4q^2}{\pi}\int_0^{\pi/4}\phi_\gamma\left(\frac{g_{\mathrm{MQAM}}}{\sin^2\theta}\right)d\theta}_{\triangleq\mathcal{I}_{2,\mathrm{MQAM}}} \quad (11)$$

$$P_s(E) = \frac{\phi_\gamma(g_{\mathrm{MDPSK}})\cos\left(\frac{\pi}{2M}\right)}{\pi}\int_0^1 t^{-1/2}\left(1-\frac{\cos\left(\frac{\pi}{M}\right)}{1+\frac{g_{\mathrm{MDPSK}}\overline{\gamma}}{m}}t\right)^{-m}\left(1-t\cos\left(\frac{\pi}{M}\right)\right)^m\left(1-t\cos^2\left(\frac{\pi}{2M}\right)\right)^{-1/2}dt$$
$$= \frac{2\phi_\gamma(g_{\mathrm{MDPSK}})\cos\left(\frac{\pi}{2M}\right)}{\pi}F_D^{(3)}\left(\frac{1}{2},m,-m,\frac{1}{2};\frac{3}{2};\frac{\cos\left(\frac{\pi}{M}\right)}{1+\frac{g_{\mathrm{MDPSK}}\overline{\gamma}}{m}},\cos\left(\frac{\pi}{M}\right),\cos^2\left(\frac{\pi}{2M}\right)\right) \quad (14)$$

where $q = 1-1/\sqrt{M}$ and $g_{\mathrm{MQAM}} = 3/(2(M-1))$. Considering the similarity of $\mathcal{I}_{1,\mathrm{MQAM}}$ to (4) and making the change of variable $t = 1-\tan^2\theta$ in $\mathcal{I}_{2,\mathrm{MQAM}}$, after some manipulations, we obtain

$$\begin{aligned} P_s(E) = &\frac{2q\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(m+1)}\phi_\gamma(g_{\mathrm{MQAM}}) \\ &\cdot {}_2F_1\left(m,\frac{1}{2};m+1;\frac{1}{1+\frac{g_{\mathrm{MQAM}}\overline{\gamma}}{m}}\right) \\ &- \frac{2q^2}{\pi(2m+1)}\phi_\gamma(2g_{\mathrm{MQAM}}) \\ &\cdot F_1\left(1,m,1;m+\frac{3}{2};\frac{1+\frac{g_{\mathrm{MQAM}}\overline{\gamma}}{m}}{1+\frac{2g_{\mathrm{MQAM}}\overline{\gamma}}{m}},\frac{1}{2}\right). \end{aligned} \quad (12)$$

Using the identity (18) along with

$$\sin^{-1}(x) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right), \quad x^2 < 1$$

we see that (12) reduces to [8, eq. (8.106)] for Rayleigh fading.

E. M-ary Differential Phase-Shift Keying (MDPSK)

The average SER for differentially coherent detection of MDPSK signals is given by [7]

$$P_s(E) = \frac{1}{\pi}\int_0^{\pi-(\pi/M)}\phi_\gamma\left(\frac{\sin^2\left(\frac{\pi}{M}\right)}{1+\cos\left(\frac{\pi}{M}\right)\cos\theta}\right)d\theta. \quad (13)$$

By change of the variable $t = \sin^2\theta/\sin^2(\pi/2-\pi/(2M))$, after some manipulations, (13) can be expressed in closed form as (14), shown at the top of the page, where $g_{\mathrm{MDPSK}} = \sin^2(\pi/M)/(1+\cos(\pi/M)) = 2\sin^2(\pi/(2M))$ and $F_D^{(n)}$ is the Lauricella function [11, eq. (2.3.6)]. If the number of variables n is equal to two, $F_D^{(n)}$ reduces to F_1, and if $n = 1$, $F_D^{(n)}$ becomes ${}_2F_1$. Note that for the special case $M = 2$ (binary DPSK) wherein $g_{\mathrm{MDPSK}} = 1$, using $F_D^{(3)}(a,b_1,b_2,b_3;c;0,0,x) = {}_2F_1(a,b_3;c;x)$ along with the identity $\sin^{-1}(z) = z\,{}_2F_1\left(1/2,1/2;3/2;z^2\right)$ [10, eq. 2.8.(13)], we see that (14) reduces to [8, eq. (8.183)]. Furthermore, for Rayleigh fading, using (21) together with simple trigonometric manipulations, we can reduce (14) to [3, eq. (8)].

APPENDIX

In this appendix, we derive the reduction formulas for $F_1(a,1,1;c;x,y)$ and $F_D^{(3)}(1/2,1,-1,1/2;3/2;x_1,x_2,x_3)$.

For $|x| < 1$ and $|y| < 1$, we have

$$F_1(a,1,1;c;x,y) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \cdot \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-1}(1-yt)^{-1}dt. \quad (15)$$

Using the partial fraction

$$\frac{1}{(1-xt)(1-yt)} = \frac{x}{(x-y)(1-xt)} + \frac{y}{(y-x)(1-yt)}$$

(15) can be written as

$$F_1(a,1,1;c;x,y) = \left(\frac{x}{x-y}\right){}_2F_1(1,a;c;x) + \left(\frac{y}{y-x}\right){}_2F_1(1,a;c;y). \quad (16)$$

Using [10, eq. 5.11.(3)] and (16), we can also reduce $F_1(a,1,b';b'+2;x,y)$ to the Gauss hypergeometric functions, such as

$$\begin{aligned} F_1&(a,1,b';b'+2;x,y) \\ &= (1-y)^{-a}\left\{\left(\frac{x-y}{x}\right){}_2F_1\left(1,a;b'+2;\frac{y-x}{y-1}\right) + \left(\frac{y}{x}\right){}_2F_1\left(1,a;b'+2;\frac{y}{y-1}\right)\right\}. \end{aligned} \quad (17)$$

A. Special Cases of Interest

1) From (16) with $a = 1$ and $c = 5/2$, we have for $|x| < 1$ and $|y| < 1$

$$F_1\left(1,1,1;\frac{5}{2};x,y\right) = \frac{3}{x-y}\left\{\sqrt{\frac{1-y}{y}}\sin^{-1}(\sqrt{y}) - \sqrt{\frac{1-x}{x}}\sin^{-1}(\sqrt{x})\right\}. \quad (18)$$

2) From (17) with $a = 1/2$ and $b' = -1$, we have for $|x| < 1$ and $|y| < 1$

$$F_1\left(\frac{1}{2},1,-1;1;x,y\right) = \frac{x-y}{x\sqrt{1-x}} + \frac{y}{x}. \quad (19)$$

3) From (17) with $a = 1/2$ and $b' = -1/2$, we have for $|x| < 1$, $|y| < 1$, and $x < y$

$$F_1\left(\frac{1}{2}, 1, -\frac{1}{2}; \frac{3}{2}; x, y\right) = \frac{\sqrt{y}}{x}\tan^{-1}\left(\sqrt{\frac{y}{1-y}}\right) - \frac{\sqrt{y-x}}{x}\tan^{-1}\left(\sqrt{\frac{y-x}{1-y}}\right). \tag{20}$$

Using (17) and [10, eq. 5.10.(1)], we have for $|x_1| < 1$, $|x_2| < 1$, and $|x_3| < 1$

$$\begin{aligned} F_D^{(3)}&\left(\frac{1}{2}, 1, -1, \frac{1}{2}; \frac{3}{2}; x_1, x_2, x_3\right) \\ &= F_1\left(\frac{1}{2}, 1, \frac{1}{2}; \frac{3}{2}; x_1, x_3\right) - \frac{x_2}{3}F_1\left(\frac{3}{2}, 1, \frac{1}{2}; \frac{5}{2}; x_1, x_3\right) \\ &= \frac{x_1 - x_2}{x_1\sqrt{x_3 - x_1}}\tan^{-1}\left(\sqrt{\frac{x_3 - x_1}{1 - x_3}}\right) \\ &\quad + \frac{x_2}{x_1\sqrt{x_3}}\tan^{-1}\left(\sqrt{\frac{x_3}{1 - x_3}}\right). \end{aligned} \tag{21}$$

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