Saddlepoint Approximation to the Outage Capacity of MIMO Channels

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***Abstract*— We put forth a saddlepoint approximation for the outage capacity of multiple-input multiple-output (MIMO) systems using the exact moment generating function of the capacity. We consider both uncorrelated and spatially correlated Rayleigh-fading channels. Our results show that the saddlepoint method gives a remarkably accurate approximation to the outage capacity even at extremely low outage probabilities.**

***Index Terms*— Channel capacity, multiple-input multiple-output (MIMO) system, outage capacity, Rayleigh fading, saddlepoint approximation (SPA), spatial fading correlation.**

I. Introduction

SINCE the seminal work of [1], [2], [3] in the area of multiple-antenna communications, much attention has been given in recent years to multiple-input multiple-output (MIMO) systems for wireless communications. A better understanding of the fundamental limit to achievable rates over time-varying MIMO channels can be obtained by investigating its outage capacity (i.e., capacity versus outage probability) in addition to ergodic (or mean) capacity [2], [3]. This requires complete statistical knowledge of a capacity random variable.

Recently, explicit determinantal formulas for the characteristic function (CF) or moment generating function (MGF) of the capacity have been derived for independent and identically distributed (i.i.d.) [4], "one-sided correlated" [5], [6], and "doubly correlated" [7], [8] Rayleigh-fading MIMO channels.[1] The analytical framework developed in these papers relies on statistical and mathematical theory: the distribution theory of complex random matrices and the theory of hypergeometric functions of matrix arguments. Using the determinantal CF formula of MIMO capacity, all statistical moments including the relevant first moment (ergodic capacity) have been obtained as trace formulas in a closed form [7].

However, a direct expression for the capacity distribution or equivalently the outage capacity is generally unknown (even for i.i.d. MIMO channels), due to the analytical difficulty in performing the inverse Fourier transform of the capacity CF.[2] Although numerical integration for the Fourier inversion formula can be carried out with high accuracy, it often converges very slowly. In [5] and [7], the fast Fourier transform (FFT) was used to efficiently evaluate the Fourier inversion integral to the capacity distribution. However, the use of the FFT introduces two types of errors: a sampling error due to the evaluation of the distribution only at discrete points, and a truncation error due to the use of only a finite number of samples (see [10] for more details on these errors). These errors are more significant in the far tail region of the distribution. A common approach to the evaluation of the outage capacity is to use a Gaussian approximation (GA) based on the exact mean and variance (see, for example, [4], [11] for i.i.d. channels and [6] for one-sided correlated channels). However, such an approximation is inaccurate at low outage probabilities.

In this letter, we put forth an accurate and efficient method, based on the saddlepoint approximation (SPA) [12], [13], [14], to calculate the outage capacity for both i.i.d. and doubly correlated Rayleigh-fading MIMO channels. We consider the so-called multiple element transmit receive antennas (METRA) model based on realistic channel measurements [15], [16] as well as a classical exponential correlation model. The saddlepoint method provides an accurate approximation to the outage capacity even at low outage probabilities, owing to the fact that the entire cumulant generating function (CGF) of the capacity is used to readjust the saddlepoint at each value of MIMO capacity in optimizing the fit of the approximation. The required computation for determining the saddlepoint is known to be a well-conditioned problem due to the convexity of the CGF [12].

II. System Model and Channel Capacity

We consider a point-to-point frequency-flat fading communication link with n_T transmit and n_R receive antennas. Let $\mathbf{H} \in \mathbb{C}^{n_\mathrm{R}\times n_\mathrm{T}}$ be a random channel matrix whose (i,j)th entries H_{ij}, $i = 1,2,\ldots,n_\mathrm{R}$, $j = 1,2,\ldots,n_\mathrm{T}$, are complex propagation coefficients between the jth transmit and the ith receive antennas with $\mathbb{E}\left\{|H_{ij}|^2\right\} = 1$. For Rayleigh fading in

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[1] "One-sided correlated" refers to the case with spatial fading correlation either at the transmitter or at the receiver, whereas "doubly correlated" refers to a more general situation where spatial fading correlation exists at both ends.

[2] As a partial solution to this problem, the capacity distribution has been derived in [9] for small numbers of antennas (i.e., $\min\{n_\mathrm{T}, n_\mathrm{R}\} \le 3$ where n_T and n_R are the numbers of transmit and receive antennas, respectively).

the presence of spatial correlation at both the transmitter and the receiver, the channel matrix $\mathbf{H}$ can be written as [17]

$$\mathbf{H} = \mathbf{\Psi}_{\mathrm{R}}^{1/2}\boldsymbol{\mathcal{H}}\mathbf{\Psi}_{\mathrm{T}}^{1/2} \tag{1}$$

where $\boldsymbol{\mathcal{H}}$ is the $n_{\mathrm{R}} \times n_{\mathrm{T}}$ matrix with i.i.d. zero-mean unit-variance complex Gaussian entries and $\mathbf{\Psi}_{\mathrm{T}}$ and $\mathbf{\Psi}_{\mathrm{R}}$ are $n_{\mathrm{T}} \times n_{\mathrm{T}}$ transmit and $n_{\mathrm{R}} \times n_{\mathrm{R}}$ receive correlation matrices, respectively.

When the receiver has perfect channel knowledge and the transmitter is constrained in its total average power with equal power allocation to each of transmit antennas, the channel capacity is described by a random variable [2], [3]

$$C = \log_2 \det\left(\mathbf{I}_{n_{\mathrm{R}}} + \frac{\eta}{n_{\mathrm{T}}}\mathbf{H}\mathbf{H}^{\dagger}\right) \qquad \text{bits/s/Hz} \tag{2}$$

where $\dagger$ denotes a complex conjugate of a matrix and η is the average signal-to-noise ratio (SNR) at each receive antenna.

When the channel is ergodic, an operational capacity measure is ergodic capacity, $\langle C\rangle = \mathbb{E}\{C\}$, obtained by averaging C over all possible channel realizations. Although $\langle C\rangle$ is the maximum achievable rate in the sense of Shannon (i.e., without delay constraints), this alone is insufficient to completely characterize the limit on the information rates that can be supported by a time-varying channel. When a stringent delay constraint is present, the notion of outage capacity provides a more meaningful measure than the ergodic capacity [2], [3]. The outage capacity at outage probability $\mathcal{P}$ is defined as

$$C_{\mathcal{P}} \triangleq \sup_{\mathcal{R}\geq 0}\left\{\mathcal{R} : \Pr\left[\log_2 \det\left(\mathbf{I}_{n_{\mathrm{R}}} + \frac{\eta}{n_{\mathrm{T}}}\mathbf{H}\mathbf{H}^{\dagger}\right) \leq \mathcal{R}\right] \leq \mathcal{P}\right\} \tag{3}$$

which implies that $F_C(C_{\mathcal{P}}) = \mathcal{P}$, where $F_C(\cdot)$ is the cumulative distribution function of C. In particular, zero-outage capacity (outage capacity at $\mathcal{P} = 0$) is known as delay-limited capacity.

III. Saddlepoint Method

When the first few moments of a random variable X are known, a Gram–Charlier or Edgeworth expansion can be used to approximate its density function [18]. These approximations are often satisfactory and allow additional flexibility compared to a GA, as they naturally introduce the skewness and kurtosis of X. However, these methods have the notorious drawback of yielding negative values in the tail regions of the distribution [18]. In contrast, when the CF or MGF of X is known, the saddlepoint method can be applied to approximate the density function [12] or the distribution function [14]. The SPA gives an accurate approximation over the whole range of X, even in the far tail of the distribution.

Let $\Phi_C(s) \triangleq \mathbb{E}\left\{e^{sC}\right\}$ and $\mathcal{K}_C(s) \triangleq \ln\Phi_C(s)$ be the MGF and CGF of C, respectively. Then, using the Lugannani–Rice formula [14], the capacity distribution can be approximated as

$$F_C(\mathcal{R}) \approx 1 - Q(u(\hat{s})) + \frac{e^{-u^2(\hat{s})/2}}{\sqrt{2\pi}}\left\{\frac{1}{u(\hat{s})} - \frac{1}{v(\hat{s})}\right\} \tag{4}$$

with

$$u(\hat{s}) = \operatorname{sgn}(\hat{s})\sqrt{2\left(\hat{s}\mathcal{R} - \mathcal{K}_C(\hat{s})\right)} \tag{5}$$

$$v(\hat{s}) = \hat{s}\sqrt{\mathcal{K}_C''(\hat{s})} \tag{6}$$

where $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-z^2/2}dz$ is the Gaussian tail integral and $\hat{s}$, called the saddlepoint, is the solution of the saddlepoint equation $\mathcal{K}_C'(\hat{s}) = \mathcal{R}$. Note that the solution to the saddlepoint equation is unique due to the convexity of the CGF. The function $\operatorname{sgn}(\hat{s})$ is defined by

$$\operatorname{sgn}(\hat{s}) = \begin{cases} +1, & \hat{s} > 0 \\ 0, & \hat{s} = 0 \\ -1, & \hat{s} < 0 \end{cases}$$

and $\mathcal{K}_C'(\cdot)$, $\mathcal{K}_C''(\cdot)$ denote the first and second derivatives of $\mathcal{K}_C(\cdot)$, respectively. The Lugannani–Rice formula for the capacity distribution in (4) has a singular point at $\mathcal{R} = \langle C\rangle$ (or equivalently $\hat{s} = 0$). At this point, (4) can be replaced by [13]

$$\begin{aligned} F_C(\langle C\rangle) &\approx \frac{1}{2} + \frac{\mathcal{K}_C'''(0)}{6\sqrt{2\pi\left[\mathcal{K}_C''(0)\right]^3}} \\ &= \frac{1}{2} + \frac{\beta_{1,C}}{6\sqrt{2\pi}} \end{aligned} \tag{7}$$

where $\mathcal{K}_C'''(\cdot)$ denotes the third derivative of $\mathcal{K}_C(\cdot)$ and

$$\beta_{1,C} \triangleq \frac{\mathbb{E}\left\{(C - \langle C\rangle)^3\right\}}{\left(\mathbb{V}\mathrm{ar}\{C\}\right)^{3/2}} \tag{8}$$

is the skewness of C. It was shown in [7] that for a symmetric antenna case (i.e., $n_{\mathrm{T}} = n_{\mathrm{R}}$), the skewness in a high-SNR regime is bounded by

$$\frac{-12\sqrt{6}\,\zeta(3)}{\pi^3} \leq \beta_{1,C} < 0 \tag{9}$$

where $\zeta(3) \approx 1.2020569$ is Apéry's constant.

A. I.I.D. MIMO Channels

Let us denote $n_{\mathrm{S}} = \min\{n_{\mathrm{T}}, n_{\mathrm{R}}\}$ and $n_{\mathrm{L}} = \max\{n_{\mathrm{T}}, n_{\mathrm{R}}\}$. For i.i.d. Rayleigh fading (i.e., $\mathbf{\Psi}_{\mathrm{T}} = \mathbf{I}_{n_{\mathrm{T}}}, \mathbf{\Psi}_{\mathrm{R}} = \mathbf{I}_{n_{\mathrm{R}}}$), the MGF of C is given by [5, eq. (25)]

$$\Phi_C(s) = K_{\mathrm{iid}}^{-1}\det\left\{\mathbf{\Omega}(s)\right\} \tag{10}$$

where

$$K_{\mathrm{iid}} = \prod_{\ell=1}^{n_{\mathrm{S}}}(n_{\mathrm{L}} - \ell)!\,(\ell - 1)! \tag{11}$$

and $\mathbf{\Omega}(s)$ is the $n_{\mathrm{S}} \times n_{\mathrm{S}}$ Hankel matrix with (i, j)th entry

$$\begin{aligned} \{\mathbf{\Omega}(s)\}_{i,j} &= \int_0^{\infty}\left(1 + \frac{\eta z}{n_{\mathrm{T}}}\right)^{s\log_2 e} z^{n_{\mathrm{L}}-n_{\mathrm{S}}+i+j-2}e^{-z}dz \\ &= (n_{\mathrm{L}} - n_{\mathrm{S}} + i + j - 2)! \\ &\quad \times {}_2F_0\left(n_{\mathrm{L}} - n_{\mathrm{S}} + i + j - 1, -s\log_2 e; -\frac{\eta}{n_{\mathrm{T}}}\right) \end{aligned} \tag{12}$$

and ${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z)$ is the generalized hypergeometric function [19, eq. (9.14.1)].

For any invertible matrix $\mathbf{G}(s)$ whose elements are differentiable with respect to s, it follows from [7, Lemma 1] that

$$\frac{d^{\ell}\ln\det\mathbf{G}(s)}{ds^{\ell}} = \operatorname{tr}\left\{\mathbf{G}_{[1]}^{(\ell-1)}(s)\right\} \tag{13}$$

$$\left\{\frac{d^\ell \mathbf{\Omega}(s)}{ds^\ell}\right\}_{i,j} = (\log_2 e)^\ell \int_0^\infty \left(1+\frac{\eta z}{n_\mathrm{T}}\right)^{s\log_2 e} \ln^\ell\left(1+\frac{\eta z}{n_\mathrm{T}}\right) z^{n_\mathrm{L}-n_\mathrm{S}+i+j-2} e^{-z} dz$$
$$= \frac{(\log_2 e)^\ell\, \ell!\, e^{n_\mathrm{T}/\eta}}{(\eta/n_\mathrm{T})^{n_\mathrm{L}-n_\mathrm{S}+i+j-1}} \sum_{k=0}^{n_\mathrm{L}-n_\mathrm{S}+i+j-2}\left[(-1)^{n_\mathrm{L}-n_\mathrm{S}+i+j-k-2}\binom{n_\mathrm{L}-n_\mathrm{S}+i+j-2}{k}\right.$$
$$\left.\times\left(\frac{\eta}{n_\mathrm{T}}\right)^{s+k+1} G_{\ell+1,\ell+2}^{\ell+2,0}\left(\frac{n_\mathrm{T}}{\eta}\,\middle|\, \begin{matrix}\overbrace{1,1,\ldots,1}^{\ell+1 \text{ ones}} \\ \underbrace{0,0,\ldots,0}_{\ell+1 \text{ zeros}}, s\log_2 e + k + 1\end{matrix}\right)\right] \tag{17}$$

TABLE I

1% OUTAGE CAPACITY $C_{0.01}$ IN BITS/S/HZ AND ITS APPROXIMATIONS FOR I.I.D. AND EXPONENTIALLY CORRELATED ($\rho_\mathrm{T} = 0.5$, $\rho_\mathrm{R} = 0.7$) MIMO CHANNELS AT $\eta = 15$ dB

Number of antennas	Correlation	Exact	Saddlepoint approximation	Gaussian approximation
$n_\mathrm{T} = n_\mathrm{R} = 2$	i.i.d.	4.538	4.524	4.456
	$\rho_\mathrm{T} = 0.5,\ \rho_\mathrm{R} = 0.7$	3.869	3.866	3.935
$n_\mathrm{T} = n_\mathrm{R} = 3$	i.i.d.	8.525	8.522	8.433
	$\rho_\mathrm{T} = 0.5,\ \rho_\mathrm{R} = 0.7$	7.093	7.094	7.145
$n_\mathrm{T} = n_\mathrm{R} = 4$	i.i.d.	12.532	12.532	12.457
	$\rho_\mathrm{T} = 0.5,\ \rho_\mathrm{R} = 0.7$	10.268	10.269	10.317
$n_\mathrm{T} = n_\mathrm{R} = 5$	i.i.d.	16.548	16.549	16.491
	$\rho_\mathrm{T} = 0.5,\ \rho_\mathrm{R} = 0.7$	13.425	13.425	13.463

where $\mathrm{tr}(\cdot)$ denotes a trace operator of a square matrix and $\mathbf{G}_{[1]}^{(\ell-1)}(s)$ is the $(\ell-1)$th derivative of the *dimatrix* $\mathbf{G}_{[1]}(s)$.[3] From (4), (10), and (13), the SPA to the capacity distribution for i.i.d. Rayleigh-fading MIMO channels is given by

$$F_C(\mathcal{R}) \approx 1 - Q\left(\mathrm{sgn}(\hat{s})\sqrt{2(\hat{s}\mathcal{R} - \ln\det\mathbf{\Omega}(\hat{s}) + \ln K_\mathrm{iid})}\right) + \frac{1}{\sqrt{2\pi}}\exp\left\{-\hat{s}\mathcal{R} + \ln\det\mathbf{\Omega}(\hat{s}) - \ln K_\mathrm{iid}\right\} \times \left\{\frac{\mathrm{sgn}(\hat{s})}{\sqrt{2(\hat{s}\mathcal{R} - \ln\det\mathbf{\Omega}(\hat{s}) + \ln K_\mathrm{iid})}} - \frac{1}{\hat{s}\sqrt{\mathrm{tr}\left\{\mathbf{\Omega}_{[1]}^{(1)}(\hat{s})\right\}}}\right\} \tag{14}$$

where $\hat{s}$ is the solution of the saddlepoint equation

$$\mathrm{tr}\left\{\mathbf{\Omega}_{[1]}(\hat{s})\right\} = \mathcal{R} \tag{15}$$

[3] The ℓth *polymatrix* of $\mathbf{G}(s)$ with respect to s is defined as [7, Definition 2]

$$\mathbf{G}_{[\ell]}(s) \triangleq \mathbf{G}^{-1}(s)\frac{d^\ell \mathbf{G}(s)}{ds^\ell}$$

and particularly, $\mathbf{G}_{[1]}(s)$ is called the dimatrix of $\mathbf{G}(s)$. Note that the well-known rule for the ℓth derivative of the determinant of an $n \times n$ matrix requires the calculations of determinants as many as the distinct terms in the multinomial series expansion of $(x_1 + x_2 + \ldots + x_n)^\ell$ [20]. For example, $\binom{n}{1}$, $\binom{n}{1} + \binom{n}{2}$, and $\binom{n}{1} + \binom{n}{1}\binom{n-1}{1} + \binom{n}{3}$ determinants are required to calculate for the first, second, and third order derivatives, respectively. In contrast, (13) requires only calculating the trace of a matrix determined by the first ℓ polymatrices $\mathbf{G}_{[\ell]}(s)$ using the relations between the polymatrices and the derivatives of the dimatrix [7, Lemma 1]. The second, third, and fourth order logarithmic derivatives of a determinant are given in [7, eqs. (14)–(16)], for example.

which can be solved numerically. At $\mathcal{R} = \langle C \rangle = \mathrm{tr}\left\{\mathbf{\Omega}_{[1]}(0)\right\}$, it follows from (7), (10), and (13) that

$$F_C(\langle C \rangle) \approx \frac{1}{2} + \frac{\mathrm{tr}\left\{\mathbf{\Omega}_{[1]}^{(2)}(0)\right\}}{6\sqrt{2\pi\,\mathrm{tr}^3\left\{\mathbf{\Omega}_{[1]}^{(1)}(0)\right\}}}. \tag{16}$$

The calculations of (14)–(16) require determining the $n_\mathrm{S} \times n_\mathrm{S}$ polymatrices $\mathbf{\Omega}_{[\ell]}(s)$ together with $\mathbf{\Omega}(s)$. The (i,j)th entry of the ℓth derivative $d^\ell\mathbf{\Omega}(s)/ds^\ell$ of $\mathbf{\Omega}(s)$ is generally given by (17) at the top of the page, where $G_{p,q}^{m,n}(\cdot)$ is the Meijer G-function [19, eq. (9.301)].[4]

B. Doubly Correlated MIMO Channels

Let us denote, for convenience,

$$(\mathbf{\Psi}_\mathrm{S}, \mathbf{\Psi}_\mathrm{L}) = \begin{cases}(\mathbf{\Psi}_\mathrm{R}, \mathbf{\Psi}_\mathrm{T}), & \text{if } n_\mathrm{R} \le n_\mathrm{T} \\ (\mathbf{\Psi}_\mathrm{T}, \mathbf{\Psi}_\mathrm{R}), & \text{otherwise}\end{cases}$$

and let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{n_\mathrm{S}}$ and $0 < \mu_1 < \mu_2 < \cdots < \mu_{n_\mathrm{L}}$ be distinct ordered eigenvalues of $\mathbf{\Psi}_\mathrm{S}$ and $\mathbf{\Psi}_\mathrm{L}$, respectively. Then, the MGF of C for doubly correlated Rayleigh-fading MIMO channels is given by [7, Theorem 1]

$$\Phi_C(s) = K_\mathrm{cor}^{-1}\Upsilon_{n_\mathrm{S}}(s)\det\Bigg(\underbrace{\begin{bmatrix}\mathbf{\Lambda}_1 \\ \mathbf{\Lambda}_2(s)\end{bmatrix}}_{=\mathbf{\Lambda}(s)}\Bigg) \tag{18}$$

[4] The generalized hypergeometric function ${}_pF_q$ and the Meijer G-function $G_{p,q}^{m,n}$ are provided as the library functions in a common mathematical software package such as MATHEMATICA or MAPLE.

TABLE II

1% OUTAGE CAPACITY $C_{0.01}$ IN BITS/S/HZ AND ITS APPROXIMATIONS FOR 4×4 MIMO CHANNELS WITH THE METRA CORRELATION MODEL [15], [16] AT $\eta = 15$ dB

Environment	Exact	Saddlepoint approximation	Gaussian approximation
i.i.d.	12.532	12.532	12.457
Picocell partially decorrelated [15, Example 1]	10.387	10.388	10.428
Microcell partially correlated [15, Example 2]	6.521	6.514	6.670
Macrocell ITU Pedestrian A [16]	6.950	6.947	7.107
Macrocell ITU Vehicular A [16]	7.295	7.296	7.415

with

$$\Upsilon_{n_{\mathrm{S}}}(s) = \prod_{\ell=1}^{n_{\mathrm{S}}-1} (s \log_2 e + \ell)^{-\ell} \tag{19}$$

$$K_{\mathrm{cor}} = \left(\frac{\eta}{n_{\mathrm{T}}}\right)^{n_{\mathrm{S}}(n_{\mathrm{S}}-1)/2} \prod_{i<j}^{n_{\mathrm{S}}} (\lambda_j - \lambda_i) \prod_{i<j}^{n_{\mathrm{L}}} (\mu_j - \mu_i) \tag{20}$$

where $\mathbf{\Lambda}_1$ is the $(n_{\mathrm{L}} - n_{\mathrm{S}}) \times n_{\mathrm{L}}$ matrix with (i,j)th entry μ_j^{i-1} and $\mathbf{\Lambda}_2(s)$ is the $n_{\mathrm{S}} \times n_{\mathrm{L}}$ matrix whose (i,j)th entry is given by

$$\begin{aligned}\{\mathbf{\Lambda}_2(s)\}_{i,j} &= \mu_j^{n_{\mathrm{L}}-n_{\mathrm{S}}-1} \int_0^\infty \left(1 + \frac{\eta \lambda_i z}{n_{\mathrm{T}}}\right)^{s \log_2 e + n_{\mathrm{S}} - 1} e^{-z/\mu_j} dz \\ &= \mu_j^{n_{\mathrm{L}}-n_{\mathrm{S}}} \, {}_2F_0 \left(1, -s \log_2 e - n_{\mathrm{S}} + 1; -\frac{\eta}{n_{\mathrm{T}}} \lambda_i \mu_j\right).\end{aligned} \tag{21}$$

Similar to (14), the SPA to the capacity distribution for doubly correlated Rayleigh-fading MIMO channels is given by at $\mathcal{R} \neq \langle C \rangle$

$$\begin{aligned}F_C(\mathcal{R}) \approx\; & 1 - Q\left(\operatorname{sgn}(\hat{s}) \sqrt{2\left(\hat{s}\mathcal{R} - \ln\left(\Upsilon_{n_{\mathrm{S}}}(\hat{s}) \det \mathbf{\Lambda}(\hat{s})\right) + \ln K_{\mathrm{cor}}\right)}\right) \\ & + \frac{1}{\sqrt{2\pi}} \exp\left\{-\hat{s}\mathcal{R} + \ln\left(\Upsilon_{n_{\mathrm{S}}}(\hat{s}) \det \mathbf{\Lambda}(\hat{s})\right) - \ln K_{\mathrm{cor}}\right\} \\ & \times \left\{ \frac{\operatorname{sgn}(\hat{s})}{\sqrt{2\left(\hat{s}\mathcal{R} - \ln\left(\Upsilon_{n_{\mathrm{S}}}(\hat{s}) \det \mathbf{\Lambda}(\hat{s})\right) + \ln K_{\mathrm{cor}}\right)}} \right. \\ & \left. - \frac{1}{\hat{s}\sqrt{\operatorname{tr}\left\{\mathbf{\Lambda}_{[1]}^{(1)}(\hat{s})\right\} + (\log_2 e)^2 \sum_{\ell=1}^{n_{\mathrm{S}}-1} \ell (\ell + \hat{s})^{-2}}} \right\}\end{aligned} \tag{22}$$

where the saddlepoint $\hat{s}$ satisfies

$$\operatorname{tr}\left\{\mathbf{\Lambda}_{[1]}(\hat{s})\right\} - (\log_2 e) \sum_{\ell=1}^{n_{\mathrm{S}}-1} \frac{\ell}{\ell + \hat{s}} = \mathcal{R}. \tag{23}$$

At $\mathcal{R} = \langle C \rangle = \operatorname{tr}\left\{\mathbf{\Lambda}_{[1]}(0)\right\} - (\log_2 e)(n_{\mathrm{S}} - 1)$, we have

$$F_C(\langle C \rangle) \approx \frac{1}{2} + \frac{\operatorname{tr}\left\{\mathbf{\Lambda}_{[1]}^{(2)}(0)\right\} - 2(\log_2 e)^3 \sum_{\ell=1}^{n_{\mathrm{S}}-1} \ell^{-2}}{6\sqrt{2\pi\left[\operatorname{tr}\left\{\mathbf{\Lambda}_{[1]}^{(1)}(0)\right\} + (\log_2 e)^2 \sum_{\ell=1}^{n_{\mathrm{S}}-1} \ell^{-1}\right]^3}}. \tag{24}$$

To compute (22)–(24), we need to determine the $n_{\mathrm{L}} \times n_{\mathrm{L}}$ polymatrices

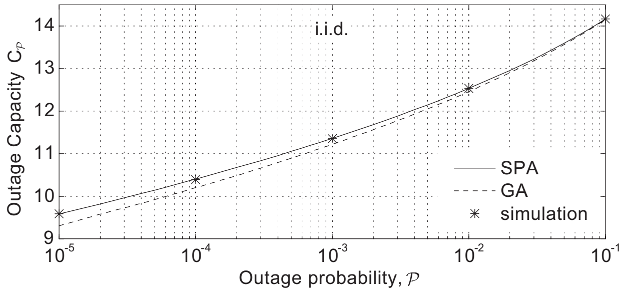
$$\mathbf{\Lambda}_{[\ell]}(s) = \begin{bmatrix} \mathbf{\Lambda}_1 \\ \mathbf{\Lambda}_2(s) \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \frac{d^\ell \mathbf{\Lambda}_2(s)}{ds^\ell} \end{bmatrix} \tag{25}$$

together with $\mathbf{\Lambda}(s)$. The (i,j)th entry of the ℓth derivative $d^\ell \mathbf{\Lambda}_2(s)/ds^\ell$ of $\mathbf{\Lambda}_2(s)$ is generally given by

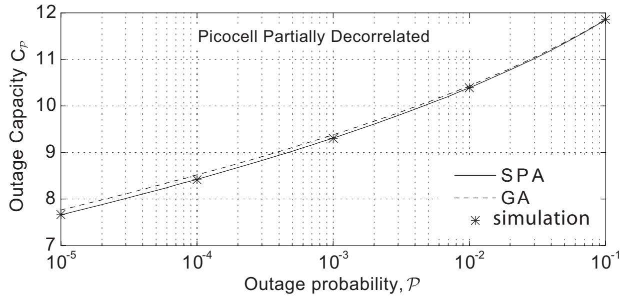
$$\begin{aligned}\left\{\frac{d^\ell \mathbf{\Lambda}_2(s)}{ds^\ell}\right\}_{i,j} &= (\log_2 e)^\ell \int_0^\infty \left(1 + \frac{\eta\lambda_i z}{n_{\mathrm{T}}}\right)^{s \log_2 e + n_{\mathrm{S}} - 1} \\ &\qquad \times \ln^\ell \left(1 + \frac{\eta \lambda_i z}{n_{\mathrm{T}}}\right) e^{-z/\mu_j} dz \\ &= (\log_2 e)^\ell \ell! \left(\frac{\eta\lambda_i}{n_{\mathrm{T}}}\right)^{s \log_2 e + n_{\mathrm{S}} - 1} \\ &\quad \times \mu_j^{s \log_2 e + n_{\mathrm{L}} - 1} \exp\left(\frac{n_{\mathrm{T}}}{\eta \lambda_i \mu_j}\right) \\ &\quad \times G_{\ell+1,\ell+2}^{\ell+2,0}\left(\frac{n_{\mathrm{T}}}{\eta\lambda_i\mu_j} \,\middle|\, \begin{matrix} 1, 1, \ldots, 1 \\ 0, 0, \ldots, 0,\; s \log_2 e + n_{\mathrm{S}} \end{matrix}\right).\end{aligned} \tag{26}$$

IV. Numerical and Simulation Results

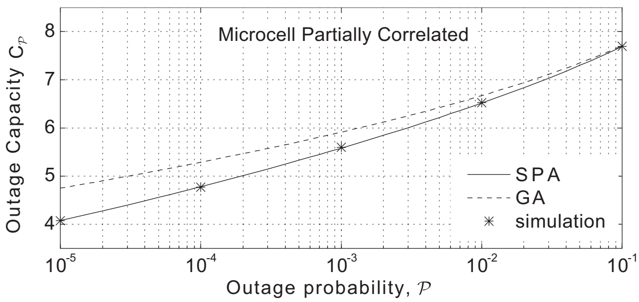
In this section, we provide some numerical results to demonstrate the accuracy of the SPA. For correlated fading, we consider the METRA model [15], [16] as well as a classical exponential correlation model in our numerical examples. The METRA model considers a separable structure of correlation at the transmit and receive sides and characterizes the correlation properties of MIMO channels using a reduced set of physical parameters such as antenna spacing, power angular spectrum, azimuth spread, and angle of arrival [15]. For exponential correlation, we set $\mathbf{\Psi}_{\mathrm{T}} = \left(\rho_{\mathrm{T}}^{|i-j|}\right)_{i,j=1,2,\ldots,n_{\mathrm{T}}}$ and $\mathbf{\Psi}_{\mathrm{R}} = \left(\rho_{\mathrm{R}}^{|i-j|}\right)_{i,j=1,2,\ldots,n_{\mathrm{R}}}$ where $\rho_{\mathrm{T}}, \rho_{\mathrm{R}} \in [0,1)$. As compared with the GA, the SPA in (14) or (22) requires the determination of the saddlepoint. This can be easily performed



(a) i.i.d.

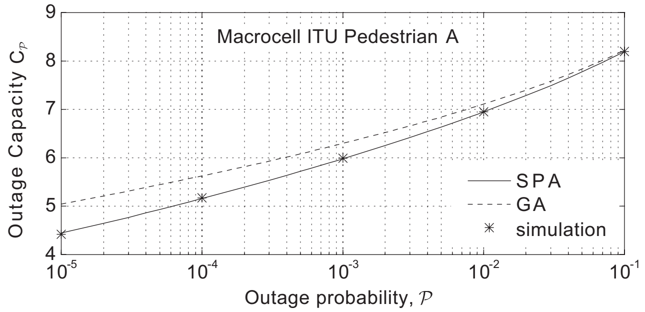


(b) Picocell partially decorellated

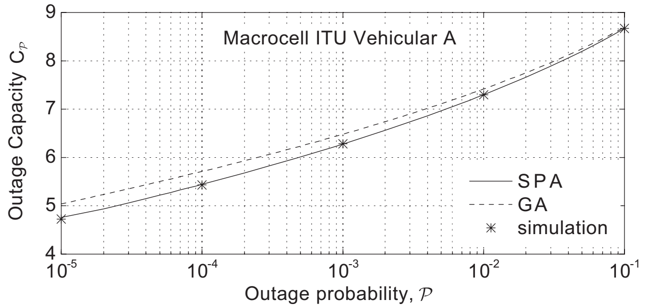


(c) Microcell partially decorellated

Fig. 1. Outage capacity $C_{\mathcal{P}}$ (bits/s/Hz) versus outage probability $\mathcal{P}$ for 4×4 MIMO channels at $\eta = 15$ dB with the METRA correlation model for (a) i.i.d. (b) picocell partially decorrelated [15, Example 1] (c) microcell partially correlated [15, Example 2] (d) macrocell ITU Pedestrian A [16] and (e) macrocell ITU Vehicular A [16] environments.



(d) Macrocell ITU Pedestrian A



(e) Macrocell ITU Vehicular A

Fig. 1. *(Continued.)* Outage capacity $C_{\mathcal{P}}$ (bits/s/Hz) versus outage probability $\mathcal{P}$ for 4×4 MIMO channels at $\eta = 15$ dB with the METRA correlation model for (a) i.i.d. (b) picocell partially decorrelated [15, Example 1] (c) microcell partially correlated [15, Example 2] (d) macrocell ITU Pedestrian A [16] and (e) macrocell ITU Vehicular A [16] environments.

using a single standard function in a common mathematical software package.

In Table I, 1% outage capacity $C_{0.01}$ in bits/s/Hz together with its SPA and GA are tabulated for both i.i.d. and exponentially correlated ($\rho_{\mathrm{T}} = 0.5$, $\rho_{\mathrm{R}} = 0.7$) MIMO channels at $\eta = 15$ dB when $n_{\mathrm{T}} = n_{\mathrm{R}} = 2$, 3, 4, and 5. The exact outage capacity is obtained by numerically evaluating the Fourier inversion formula for the capacity distribution (see, e.g., [5, eq. (32)]). We can see that the SPA is more accurate than the GA for all cases. In particular, the SPA agrees with the exact value down to two decimal places. This excellent agreement is due to the fact that the saddlepoint method uses the entire CGF beyond the mean and variance of the capacity. In Table II, 1% outage capacity $C_{0.01}$ and its SPA and GA are tabulated for 4×4 MIMO channels with correlation matrices obtained by the METRA model given in [15, p. 72] and [16]. Similar observations are made on the accuracy of approximations as those in Table I. We can also see that spatial fading correlation yields a considerable decrease in 1% outage capacity for practical MIMO channels: more than 40% reduction, except for the partially decorrelated picocell case, which is a relatively benign propagation environment (17% reduction).

To ascertain the effect of outage probability on the accuracy of the SPA and GA, Fig. 1 shows the outage capacity $C_{\mathcal{P}}$ (bits/s/Hz) as a function of outage probability $\mathcal{P}$ for 4×4 MIMO channels at $\eta = 15$ dB in five different environments: (a) i.i.d.; (b) picocell partially decorrelated; (c) microcell partially correlated; (d) macrocell ITU Pedestrian A; (e) macrocell ITU Vehicular A. In addition to the SPA and GA, Monte Carlo simulation results are also given. It can be seen that the SPA is in excellent agreement with the simulation results for all outage probability values ranging from 10^{-5} to 0.1, while the GA is shown to cause discrepancies at low outage probabilities. In Fig. 2, the SPA and Monte Carlo simulation results are compared for the 1% outage capacity $C_{0.01}$ (bits/s/Hz) versus the SNR η for exponentially correlated ($\rho_{\mathrm{T}} = 0.5$, $\rho_{\mathrm{R}} = 0.7$) MIMO channels when $n_{\mathrm{T}} = n_{\mathrm{R}} = 2$, 3, and 4. We can observe again that the SPA agrees almost exactly with the simulation results for all ranges of SNRs. In order to calculate the outage capacity (at any outage probability) by Monte Carlo simulations, we should obtain the whole cumulative distribution of the capacity. In particular, it requires exhaustive simulations at low outage probabilities. For example, at the outage probability of 10^{-5}, we need an exhaustive Monte Carlo simulation of more than $10\,000\,000$ trials for a reliable result.

V. Conclusions

We put forth the saddlepoint method to approximate the outage capacity for both i.i.d. and doubly correlated Rayleigh-

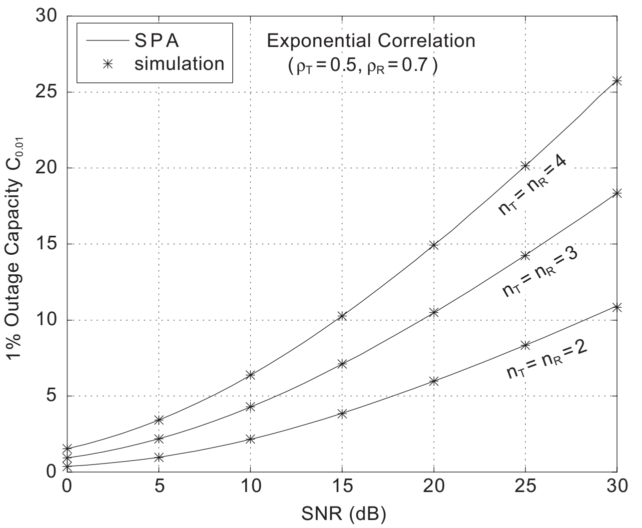


Fig. 2. 1% outage capacity $C_{0.01}$ (bits/s/Hz) versus the SNR η for exponentially correlated ($\rho_{\mathrm{T}} = 0.5$, $\rho_{\mathrm{R}} = 0.7$) MIMO channels. $n_{\mathrm{T}} = n_{\mathrm{R}} = 2$, 3, and 4.

fading MIMO channels using the MGF of the capacity and its derivatives. To illustrate our analytical results, we considered the METRA correlation model as well as the classical exponential correlation model. Our results demonstrate that the SPA provides an accurate approximation to the outage capacity even at low outage probabilities where the GA incurs discrepancies. Without an ill-conditioned numerical inversion of the MGF or CF, the outage capacity can be calculated efficiently, with sufficient accuracy, by the saddlepoint method.

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