Effect of Keyholes on the Symbol Error Rate of Space-Time Block Codes

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***Abstract*—In multiple-input multiple-output (MIMO) fading environments, degenerate channel phenomena, so-called *keyholes* or *pinholes*, may exist under realistic assumptions where a MIMO channel has uncorrelated spatial fading between antenna arrays but a rank-deficient transfer matrix. In this letter, we analyze the average symbol error rate (SER) of orthogonal space-time block codes (STBCs) with M-PSK and M-QAM constellations over keyhole channels.**

***Index Terms*—Keyhole, MIMO channels, space-time block codes (STBCs), symbol error rate (SER), transmit diversity.**

I. Introduction

RECENTLY, in multiple-input multiple-output (MIMO) fading environments, the existence of rank-deficient channels called as *keyhole* or *pinhole* channels has been proposed and demonstrated through physical examples that have uncorrelated transmit and receive signals but only have a single or reduced degree of freedom [1], [2]. This rank deficiency reduces achievable spectral efficiency and link quality in MIMO systems.

Space-time block coding is a modulation scheme for the use of multiple transmit antennas providing a simple transmit diversity scheme with the same diversity order as maximal-ratio receiver combining [3]–[5]. Due to the orthogonal structure of the space-time block code (STBC), maximum-likelihood (ML) decoding can be implemented by the single-symbol decoding based on linear processing at the receiver and it is well known that the orthogonal space-time block encoding and decoding (signal combining) transform a MIMO fading channel into an equivalent single-input single-output (SISO) Gaussian channel with a path gain of the squared Frobenius norm of the channel matrix [6]–[8].

In this letter we study the effect of keyholes on the SER of the orthogonal STBC with the assumption that the rich multiple scattering at the transmit and receive arrays may result in independent Rayleigh fading. In this case, the fading between each pair of the transmit and receive antennas in the presence of the keyhole is characterized by *double Rayleigh* fading, i.e., a product of two independent Rayleigh distributions [1]. In order to evaluate the exact average symbol error rate (SER) of the orthogonal STBC with M-PSK and M-QAM constellations over keyhole channels, we first derive the moment generating function (MGF) of a symbol signal-to-noise ratio (SNR) after space-time block decoding. Then, using the MGF-based approach for evaluating the error performance over fading channels [10], we express the average SER of the STBC in the form of single finite-range integrals involving only the derived MGF.

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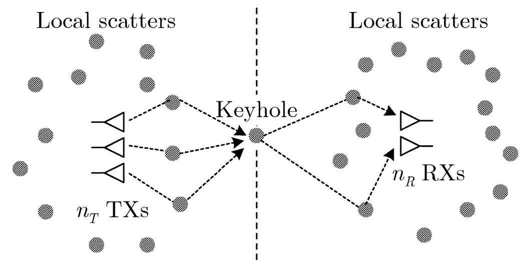

Fig. 1. Keyhole MIMO channels.

II. System Model

We consider a MIMO wireless communication system with n_T transmit and n_R receive antennas.

A. Keyhole and Channel Model

Assuming a non-line of sight and local scattering model at both transmit and receive sides for outdoor transmission, scatters are placed randomly close to either the transmit or receive array. Furthermore, the arrangement of scatters is assumed to be quasi-static, therefore the random arrangement will change at certain intervals. In general, this leads to a quasi-static, frequency-flat, and uncorrelated MIMO channel for narrowband signals. However, in realistic environments the channel degeneracy may arise due to the keyhole or pinhole effect (see for details [1], [2]), as shown in Fig. 1. The only way for the radio wave to propagate from the transmitter to the receiver is to pass through the keyhole. In this case, each entry of the channel matrix is a product of two complex Gaussian random variables (RVs), in contrast to the complex Gaussian RV normally assumed in wireless channels. The channel matrix $\mathbf{H}$ for keyhole channels is given by [1]

$$\mathbf{H} = \begin{pmatrix} \alpha_1\beta_1 & \alpha_2\beta_1 & \dots & \alpha_{n_T}\beta_1 \\ \alpha_1\beta_2 & \alpha_2\beta_2 & \dots & \alpha_{n_T}\beta_2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1\beta_{n_R} & \alpha_2\beta_{n_R} & \dots & \alpha_{n_T}\beta_{n_R} \end{pmatrix} \quad (1)$$

where $\{\alpha_j\}_{j=1}^{n_T}$ and $\{\beta_i\}_{i=1}^{n_R}$ describe the rich scattering at the transmit and receive arrays, respectively. In (1), α_j and β_i are modeled as independent zero-mean complex Gaussian RVs with unit variance and the keyhole is assumed to ideally reradiate the captured energy, like the transmit and receive scatters. Note that as all of α_j and β_i are independent, all entries of the channel matrix $\mathbf{H}$ are uncorrelated but $\mathrm{rank}\,(\mathbf{H}) = 1$.

B. Space-Time Block Codes

In this subsection, we present the basic principles of the STBC following the seminal contributions due to Tarokh *et al.* [4], [5]. K_b information bits are mapped as symbols $x_1, x_2, \ldots, x_K$, which are selected from the M-PSK or M-QAM signal constellation by Gray mapping, where $b = \log_2 M$. Then, $\{x_k\}_{k=1}^{K}$ are encoded by a space-time block code defined by a $p \times n_T$ column orthogonal transmission matrix $\mathcal{G}$ whose entries g_{ij}, $i = 1, 2, \ldots, p$, and $j = 1, 2, \ldots, n_T$, are linear combinations of $x_1, x_2, \ldots, x_K$ and their conjugates. At time k, signals $\{g_{kj}\}_{j=1}^{n_T}$, $k = 1, 2, \ldots, p$, are transmitted simultaneously through n_T transmit antennas. Since p symbol durations are necessary to transmit K symbols, the rate of the STBC is $R = K/p$.

We assume that the average energy of the symbols transmitted from each antenna is normalized to be E_s/n_T so that the average power of the received signal at each receive antenna is equal to $(E_s/n_T)\sum_{j=1}^{n_T} E\left[|H_{ij}|^2\right] = E_s$ and the average SNR per receive antenna is E_s/N_0, where N_0 is the variance of a Gaussian noise at each receive antenna.

III. PDF and MGF of the Symbol SNR After Space-Time Block Decoding

The instantaneous SNR per symbol after space-time block decoding is given by [7]

$$\gamma_{\mathrm{STBC}} = \frac{\|\mathbf{H}\|_F^2 E_s}{n_T R N_0} \tag{2}$$

where $\|\mathbf{H}\|_F^2$ is the squared Frobenius norm of $\mathbf{H}$. In the following, we derive the probability density function (pdf) and MGF of γ_{STBC}, which are used to evaluate the SER of the STBC in Section IV. To this end, we first derive the pdf and MGF of $\|\mathbf{H}\|_F^2$.

Let $Y = \|\mathbf{H}\|_F^2$, $U = \sum_{j=1}^{n_T} |\alpha_j|^2$, and $V = \sum_{i=1}^{n_R} |\beta_i|^2$. Then we can rewrite the squared Frobenius norm of the channel matrix as follows:

$$Y = \sum_{i=1}^{n_R}\sum_{j=1}^{n_T} |\alpha_j|^2 \cdot |\beta_i|^2 = \sum_{j=1}^{n_T} |\alpha_j|^2 \cdot \sum_{i=1}^{n_R} |\beta_i|^2 = U \cdot V. \tag{3}$$

Since U and V are sums of n_T and n_R independent exponential RVs, respectively, they are central chi-square distributed with $2n_T$ and $2n_R$ degrees of freedom. The pdf of Y is, therefore, given by

$$\begin{aligned} p_Y(y) &= \int_{-\infty}^{\infty} \frac{1}{|u|} p_U(u) \cdot p_V\left(\frac{y}{u}\right) du \\ &= \frac{2y^{(n_T+n_R)/2-1}}{\Gamma(n_T)\Gamma(n_R)} K_{n_R-n_T}\left(2\sqrt{y}\right), \quad y \ge 0 \end{aligned} \tag{4}$$

where $\Gamma(\cdot)$ is the gamma function and $K_\nu(\cdot)$ is the modified Hankel function of the νth order and its integral representation is given by [11, eq. (8.432.6)]

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{1}{t^{\nu+1}} e^{-t-(z^2/4t)} dt|, \qquad \arg z| < \frac{\pi}{2}, \quad \mathrm{Re}\{z^2\} > 0. \tag{5}$$

Using (5), we can find the MGF of Y as

$$\begin{aligned} \phi_Y(s) &\triangleq E\left[e^{-sY}\right] = \int_0^\infty e^{-sy} \cdot p_Y(y)\, dy \\ &= \int_0^\infty \int_0^\infty \frac{t^{n_T-n_R-1} e^{-t-(s+1/t)y} y^{n_R-1}}{\Gamma(n_T)\Gamma(n_R)} dt dy \\ &= \frac{1}{\Gamma(n_T)} \int_0^\infty t^{n_T-1} e^{-t} (1+st)^{-n_R} dt \\ &= s^{-n_T} \Psi\left(n_T, n_T - n_R + 1; s^{-1}\right) \end{aligned} \tag{6}$$

where $\Psi(a, b; z)$ is the confluent hypergeometric function [11, eq. (9.211.4)]. From the identify ${}_2F_0\left(a, b; ; -z^{-1}\right) = z^a \Psi(a, a-b+1; z)$ [12, eq. 6.6.(1)], we have

$$\phi_Y(s) = {}_2F_0(n_T, n_R; ; -s) \tag{7}$$

where ${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z)$ is the generalized hypergeometric function [11, eq. (9.14.1)], [12, eq. 4.1.(1)].

From the pdf and MGF of the squared Frobenius norm of the channel matrix in (4) and (7), the pdf and MGF of γ_{STBC} for keyhole MIMO channels, respectively, can be readily written as

$$\begin{aligned} p_{\gamma_{\mathrm{STBC}}}(\gamma) &= \frac{1}{\xi} \cdot p_Y\left(\frac{\gamma}{\xi}\right) \\ &= \frac{2\xi^{-(n_T+n_R)/2}}{\Gamma(n_T)\Gamma(n_R)} \gamma^{(n_T+n_R)/2-1} \\ &\quad \cdot K_{n_R-n_T}\left(2\sqrt{\frac{\gamma}{\xi}}\right), \quad \gamma \ge 0 \end{aligned} \tag{8}$$

$$\begin{aligned} \phi_{\gamma_{\mathrm{STBC}}}(s) &= \phi_Y(s\xi) \\ &= {}_2F_0(n_T, n_R; ; -s\xi) \end{aligned} \tag{9}$$

where $\xi = (E_s/N_0)/(n_T R)$. Note that for independent and identically distributed (i.i.d.) Rayleigh MIMO channels in the absence of the keyhole, $\|\mathbf{H}\|_F^2$ is the sum of $n_T n_R$ i.i.d. exponential RVs and then the MGF of γ_{STBC} can be easily written as $\phi_{\gamma_{\mathrm{STBC}}}(s) = (1+s\xi)^{-n_T n_R}$.

IV. Symbol Error Rates

From the MGF of γ_{STBC} in Section III, we can evaluate the average SER of the orthogonal STBC over keyhole channels using a well-known MGF-based approach for evaluating the error performance of a digital communication system over fading channels [10].

A. M-PSK

The conditional SER for coherent M-PSK signals is given by [10]

$$P_s^{\mathrm{MPSK}}(E|\gamma) = \frac{1}{\pi} \int_0^{\pi-(\pi/M)} e^{-\gamma g_{\mathrm{MPSK}}/\sin^2\theta} d\theta \tag{10}$$

where $g_{\mathrm{MPSK}} = \sin^2(\pi/M)$. Averaging (10) over the pdf $p_{\gamma_{\mathrm{STBC}}}(\gamma)$ and using (9), the average SER of space-time block codes with M-PSK over keyhole MIMO channels is given by

$$P_s^{\mathrm{MPSK}}(E) = \frac{1}{\pi} \int_0^{\pi-(\pi/M)} {}_2F_0\left(n_T, n_R; ; -\frac{\xi g_{\mathrm{MPSK}}}{\sin^2\theta}\right) d\theta. \tag{11}$$

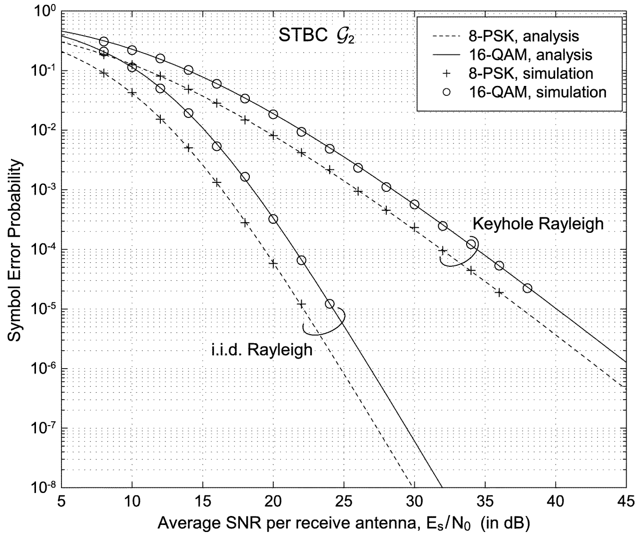

Fig. 2. SERs versus average SNR per receive antenna for the STBC $\mathcal{G}_2$ with 8-PSK and 16-QAM. $n_R = 2$.

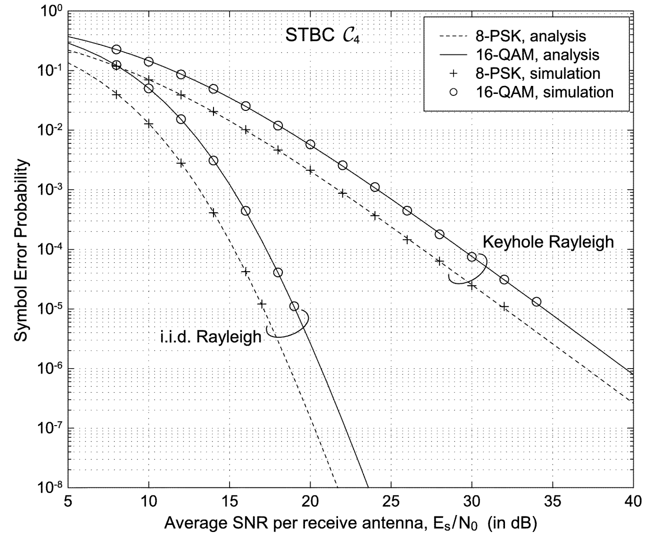

Fig. 3. SERs versus average SNR per receive antenna for the STBC $\mathcal{C}_4$ with 8-PSK and 16-QAM. $n_R = 2$.

B. M-QAM

The conditional SER for coherent square M-QAM signals is given by [10]

$$P_s^{\mathrm{MQAM}}(E|\gamma) = \frac{4q}{\pi}\int_0^{\pi/2} e^{-\gamma g_{\mathrm{MQAM}}/\sin^2\theta} d\theta - \frac{4q^2}{\pi}\int_0^{\pi/4} e^{-\gamma g_{\mathrm{MQAM}}/\sin^2\theta} d\theta \quad (12)$$

where $q = 1 - 1/\sqrt{M}$ and $g_{\mathrm{MQAM}} = 3/\{2(M-1)\}$. Similar to (11), the average SER of the STBC with M-QAM over keyhole MIMO channels can be readily shown as

$$P_s^{\mathrm{MQAM}}(E) = \frac{4q}{\pi}\int_0^{\pi/2} {}_2F_0\left(n_T, n_R; ; -\frac{\xi g_{\mathrm{MQAM}}}{\sin^2\theta}\right) d\theta - \frac{4q^2}{\pi}\int_0^{\pi/4} {}_2F_0\left(n_T, n_R; ; -\frac{\xi g_{\mathrm{MQAM}}}{\sin^2\theta}\right) d\theta. \quad (13)$$

V. Numerical Results and Discussions

We provide the results of our analysis and compare them with simulation results in order to verify the analysis. For two and four transmit antennas, we use the following one-rate STBC $\mathcal{G}_2$ (Alamouti code) and 3/4-rate STBC $\mathcal{C}_4$ given in [3] and [9], respectively

$$\mathcal{G}_2 = \begin{pmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{pmatrix} \text{ and } \mathcal{C}_4 = \begin{pmatrix} x_1 & x_2 & x_3 & 0 \\ -x_2^* & x_1^* & 0 & -x_3 \\ -x_3^* & 0 & x_1^* & x_2 \\ 0 & x_3^* & -x_2^* & x_1 \end{pmatrix}$$

where the superscript $*$ denotes the complex conjugate.

Figs. 2 and 3 show the SER versus average symbol SNR per receive antenna for the STBCs $\mathcal{G}_2$ and $\mathcal{C}_4$ with 8-PSK and 16-QAM, respectively. These results are given for two receive antennas. The transmission rates of 8-PSK and 16-QAM $\mathcal{G}_2$ codes are 3 and 4 bits/s/Hz, respectively. Since $\mathcal{C}_4$ is of 3/4-rate, the transmission rates for 8-PSK and 16-QAM are 2.25 and 3 bits/s/Hz, respectively. From these figures, we observe that the analysis agrees exactly with the simulation results and the keyhole reduces the diversity advantage (i.e., a slope of the curve) of the STBC significantly. This performance degradation is due to the fact that a keyhole channel has only single degree of freedom and will fade twice as often as a normal i.i.d. channel.

References

[1] D. Chizhik, G. J. Foschini, M. J. Gans, and R. A. Valenzuela, "Keyholes, correlations, and capacities of multielement transmit and receive antennas," *IEEE Trans. Wireless Commun.*, vol. 1, pp. 361–368, Apr. 2002.

[2] D. Gesbert, H. Bolcskei, D. A. Gore, and A. J. Paulraj, "Outdoor MIMO wireless channels: Models and performance prediction," *IEEE Trans. Commun.*, to be published.

[3] S. M. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE J. Select. Areas Commun.*, vol. 16, pp. 1451–1458, Oct. 1998.

[4] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, "Space-time block codes from orthogonal designs," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1456–1467, July 1999.

[5] ——, "Space-time block coding for wireless communications: Performance results," *IEEE J. Select. Areas Commun.*, vol. 17, pp. 451–460, Mar. 1999.

[6] X. Li, T. Luo, G. Yue, and C. Yin, "A squaring method to simplify the decoding of orthogonal space-time block codes," *IEEE Trans. Commun.*, vol. 49, pp. 1700–1703, Oct. 2001.

[7] H. Shin and J. H. Lee, "Exact symbol error probability of orthogonal space-time block codes," in *Proc. IEEE GLOBECOM'02*, Taipei, Taiwan, R.O.C., 2002.

[8] G. Bauch and J. Hagenauer, "Smart versus dump antennas—Capacities and FEC performance," *IEEE Commun. Lett.*, vol. 6, pp. 55–57, Feb. 2002.

[9] O. Tirkkonen and A. Hottinen, "Square-matrix embeddable space-time block codes for complex signal constellations," *IEEE Trans. Inform. Theory*, vol. 48, pp. 384–395, Feb. 2002.

[10] M. K. Simon and M.-S. Alouini, *Digital Communication Over Fading Channels: A Unified Approach to Performance Analysis*. New York: Wiley, 2000.

[11] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed. San Diego, CA: Academic, 2000.

[12] A. Erdelyi, *Higher Transcendental Functions*. New York: McGraw-Hill, 1953, vol. 1.