Random Coding Error Exponent for Dual-Hop Nakagami-m Fading Channels with Amplify-and-Forward Relaying

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***Abstract*—We derive the random coding error exponent for dual-hop transmission over Nakagami-m fading channels with amplify-and-forward relaying. This information-theoretic measure allows us to determine the capacity and cutoff rate as well as to gain valuable insight into the inherent tradeoff between the communication reliability and transmission rate in such channels.**

***Index Terms*—Amplify-and-forward relay, Nakagami-m fading, random coding error exponent.**

I. Introduction

COOPERATIVE relay transmission has been shown to be an effective way to provide robustness to wireless fading [1]–[7]. In such communication systems, the relay(s) would process incoming signals from the source with a certain protocol depending on the amount of channel state information (CSI), available relaying overhead, or required complexity. The relay protocols can be classified into mainly decode-and-forward and amplify-and-forward (AF) relaying. The error performance of dual-hop AF relay systems has been studied in Rayleigh and Nakagami-m fading channels [1], [2]. More recently, the opportunistic relaying has been proposed and analyzed in [3]–[5] to simply realize the diversity benefit of multiple-relay cooperation, while the (centralized) power allocation problem among AF relays has been presented in [6] under uncertainty of global CSI. The exact error probability with maximal-ratio combining has been further analyzed for dual-hop cooperative diversity systems with multiple-antenna destination reception in Nakagami-m fading channels [7].

In this letter, instead of individually considering only the achievable rate or error probability as a performance measure, we derive the random coding error exponent for dual-hop AF relay transmission over Nakagami-m fading channels. The random coding error exponent is the classical lower bound to Shannon's reliability function and gives insight into the fundamental tradeoff between the communication reliability and transmission rate [8]–[11]. Therefore, our results serve to reveal this inherent tradeoff for relay transmission and show its effectiveness in mitigating wireless fading with a positive impact on the error exponent. Moreover, the exponent analysis is general enough to obtain the closed-form expressions for the capacity and cutoff rate of such dual-hop AF relay channels.

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II. System Model

We consider a dual-hop communication system with a single AF relay. Under a half-duplex constraint, the source transmits information to the relay in the first hop and then the relay amplifies and forwards the received signal to the destination in the second hop [1]–[3]. With ideal/hypothetical AF relaying, the end-to-end signal-to-noise ratio (SNR) at the destination is given by [1], [2]

$$\gamma_{\mathrm{end}} = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} \tag{1}$$

where γ_1 and γ_2 are instantaneous SNR's of the source-to-relay and relay-to-destination links, respectively.[1]

Considering that the two links associated with γ_1 and γ_2 are subject to independent and identically distributed Nakagami-m fading, the probability density function of the end-to-end SNR γ_{end} is then given by [2]

$$p_{\gamma_{\mathrm{end}}}(\gamma) = \frac{2\sqrt{\pi}\,(m/\bar{\gamma})^m}{\Gamma^2(m)} \gamma^{m-1} e^{-4m\gamma/\bar{\gamma}} \times \Psi\left(\frac{1}{2} - m, 1 - m; \frac{4m\gamma}{\bar{\gamma}}\right), \quad \gamma \geq 0 \tag{2}$$

where $\Gamma(z)$ is the Euler's gamma function [12, 8.310.1], $\Psi(a, b; z)$ is the Tricomi's confluent hypergeometric function [12, eq. (9.211.4)], m is a parameter representing the fading severity of each link, and $\bar{\gamma}$ is the average SNR per link.

III. Random Coding Exponent Analysis

The reliability function or error exponent for a communication channel describes the best exponent decay in the error probability as a function of the codeword length [8]–[11]. Therefore, it serves to indicate a coding requirement to achieve a certain level of communication reliability at a rate below the capacity. Although finding the exact error exponent for a nontrivial channel is a challenging task, its classical lower bound, known as the random coding error exponent, is available due to Gallager by the random coding arguments.

[1]The direct source-to-destination link is assumed to be *infeasible* or to have *poor* connection, as considered in [1]–[4], [6]. The ideal/hypothetical AF relaying ignores the noise figure at the relay from the relaying gain, rendering the end-to-end SNR in an analytically more tractable form, as in (1), and leading to a tight upper bound for the exact form $\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2+1}$ of CSI-assisted AF relaying [1]. Hence, it serves to provide a benchmark for all practical AF relaying with the source-to-relay CSI.

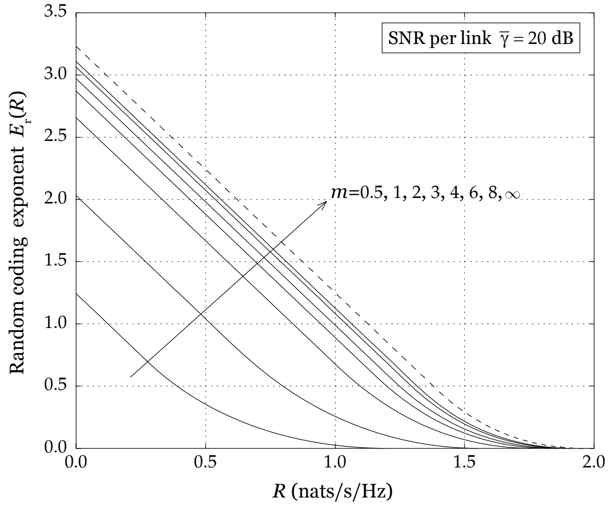

Fig. 1. Random coding error exponent versus R at $\bar{\gamma} = 20$ dB when $m = 0.5, 1, 2, 3, 4, 6, 8$, and ∞.

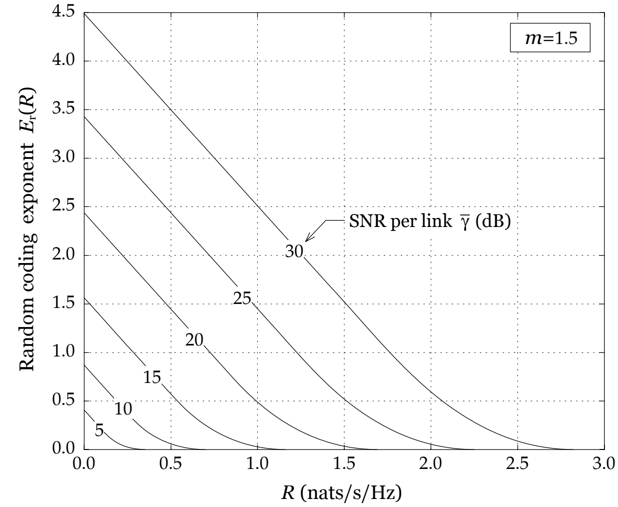

Fig. 2. Random coding error exponent versus R for $m = 1.5$ when $\bar{\gamma} = 5, 10, 15, 20, 25$, and 30 dB.

Following [8, Theorem 5.6.2], the random coding error exponent for dual-hop AF relay channels with the Gaussian input distribution can be written as

$$E_{\mathrm{r}}(R) = \max_{0 \le \rho \le 1} \{E_0(\rho) - 2\rho R\} \tag{3}$$

with

$$E_0(\rho) = -\ln \mathbb{E}_{\gamma_{\mathrm{end}}} \left\{ \left(1 + \frac{\gamma_{\mathrm{end}}}{1+\rho}\right)^{-\rho} \right\} \tag{4}$$

where R is the transmission rate in nats/s/Hz and the factor 2 of ρR in (3) is due to the dual-hop nature.[2] The maximum of the exponent over ρ occurs at $R = \frac{1}{2}\left[\partial E_0(\rho)/\partial\rho\right]\big|_{\rho=\rho^*}$ and hence, the slope of the $E_{\mathrm{r}}(R) \sim R$ curve at a rate R is equal to $-2\rho^*$. The maximizing ρ^* lies in $[0, 1]$ if

$$R_{\mathrm{cr}} = \frac{1}{2}\left[\frac{\partial E_0(\rho)}{\partial\rho}\right]\bigg|_{\rho=1} \le R \le \frac{1}{2}\left[\frac{\partial E_0(\rho)}{\partial\rho}\right]\bigg|_{\rho=0} = C \tag{5}$$

where R_{cr} is the critical rate and C is the (ergodic) capacity [8, Section 5.6]. For $R < R_{\mathrm{cr}}$, the slope of the $E_{\mathrm{r}}(R) \sim R$ curve is equal to -2 at $\rho^* = 1$, yielding $E_{\mathrm{r}}(R) = E_0(1) - 2R$.

Theorem 1: With the Gaussian input distribution, the random coding error exponent for dual-hop Nakagami-m fading channels with ideal/hypothetical AF relaying is given by

$$E_{\mathrm{r}}(R) = \max_{0 \le \rho \le 1} \{E_0(\rho) - 2\rho R\} \tag{6}$$

with $E_0(\rho)$ shown (7) at the bottom of the page, where $G_{p,q}^{m,n}(\cdot)$ is the Meijer's G-function [13, eq. (8.2.1.1)].

Proof: See Appendix A. ■

From the exponent expression, we can further deduce the capacity and cutoff rate for dual-hop AF Nakagami-m fading channels. These quantities are crucial information-theoretic measures: the capacity determines the maximum achievable rate, while the cutoff rate determines the maximum practical transmission rate for possible sequential decoding strategies.

Corollary 1: The (ergodic) capacity in nats/s/Hz for dual-hop Nakagami-m fading channels with ideal/hypothetical AF relaying is given by

$$C = \frac{\sqrt{\pi}\,(m/\bar{\gamma})^m}{\Gamma^2(m)}\, G_{3,4}^{4,1}\left(\frac{4m}{\bar{\gamma}} \,\middle|\, \begin{matrix} -m, 1-m, 1/2 \\ 0, m, -m, -m \end{matrix}\right). \tag{8}$$

Proof: See Appendix B. ■

The capacity of dual-hop AF Weibull fading channels is recently expressed also in terms of the Meijer's G-function using a simple form $\min(\gamma_1, \gamma_2)$ of the upper bound to the end-to-end SNR [14].

Corollary 2: The cutoff rate in nats/s/Hz for dual-hop Nakagami-m fading channels with ideal/hypothetical AF relaying is given by

$$R_0 = -\frac{1}{2}\ln\left[\frac{2^{m+1}\sqrt{\pi}\,(m/\bar{\gamma})^m}{\Gamma^2(m)} \times G_{2,3}^{3,1}\left(\frac{8m}{\bar{\gamma}} \,\middle|\, \begin{matrix} 1-m, 1/2 \\ 0, m, 1-m \end{matrix}\right)\right]. \tag{9}$$

[2]The variations of [8, Theorem 5.6.2] for fading channels have been well studied in the literature (see, e.g., [9]–[11] and reference therein). In particular, we arrive at (4) by taking the same steps leading to [9, eq. (9)] or by setting $\beta = 1$ in [10, eqs. (10) and (11)]. To establish a tighter random coding bound for a more general problem, Gallager further introduced one more free parameter to be optimized (see [8, Section 7.3] for details). When $\beta = 1$ (or equivalently, $r = 0$) in [10] (more generally, $\beta = n_{\mathrm{T}}$ in [11]), the problem turns back to the maximization problem, as in (3), over only a single free parameter ρ.

$$E_0(\rho) = \begin{cases} -\ln\left[\dfrac{2\sqrt{\pi}\,(m/\bar{\gamma})^m (1+\rho)^m}{\Gamma^2(m)\,\Gamma(\rho)}\, G_{2,3}^{3,1}\left(\dfrac{4m(1+\rho)}{\bar{\gamma}} \,\middle|\, \begin{matrix} 1-m, 1/2 \\ 0, m, \rho-m \end{matrix}\right)\right], & \text{for } 0 < \rho \le 1 \\ 0, & \text{for } \rho = 0 \end{cases} \tag{7}$$

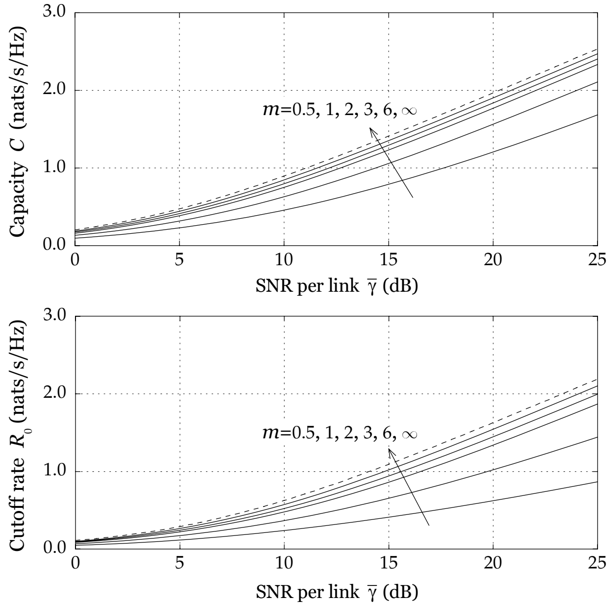

Fig. 3. Capacity and cutoff rate in nats/s/Hz versus $\bar{\gamma}$ when $m = 0.5, 1, 2, 3, 6$, and ∞.

Proof: It follows immediately from (7) by setting $\rho = 1$ and the fact that $R_0 = E_0(1)/2$. ■

IV. Numerical Results and Discussion

Fig. 1 shows the random coding error exponent versus a rate R at the average SNR per link $\bar{\gamma} = 20$ dB when the Nakagami fading parameter $m = 0.5, 1, 2, 3, 4, 6, 8$, and ∞. Fig. 2 shows the random coding error exponent versus a rate R for $m = 1.5$ when $\bar{\gamma} = 5, 10, 15, 20, 25$, and 30 dB. It can be seen from the figures that the error exponent at rates below the capacity increases with the fading parameter m or the SNR $\bar{\gamma}$, indicating that the error probability at fixed R decreases (or the same level of communication reliability is achievable at a higher rate R) as the channel condition improves. For example, as $m \to \infty$, the end-to-end SNR becomes $\gamma_{\text{end}} \to \frac{1}{2}\bar{\gamma}$ and the dual-hop channel eventually behaves like an AWGN channel. At the rate $R = 0.5$ (nats/s/Hz), the exponents are equal to 1.04 and 2.24 for dual-hop Rayleigh fading ($m = 1$) and AWGN ($m \to \infty$) cases, respectively. These values reveal that more than twice the codeword length is required for dual-hop Rayleigh fading to achieve the same communication reliability at $R = 0.5$ as the AWGN case (see [10], [11] for the connection between the error exponent and coding complexity). In Fig. 3, the capacity and cutoff rate in nats/s/Hz versus $\bar{\gamma}$ are depicted for $m = 0.5, 1, 2, 3, 6$, and ∞, showing that the relay transmission enables us to achieve these rates even when the direct communication between the source and destination is infeasible.

Appendix

A. Proof of Theorem 1

Since it is obvious from (4) that $E_0(\rho) = 0$ at $\rho = 0$, we derive $E_0(\rho)$ for $0 < \rho \le 1$. From (2) and (4), we have

$$E_0(\rho) = -\ln\left[\frac{2\sqrt{\pi}\,(m/\bar{\gamma})^m}{\Gamma^2(m)\,\Gamma(\rho)} \int_0^\infty \gamma^{m-1} G_{1,1}^{1,1}\left(\frac{\gamma}{1+\rho}\,\middle|\, \begin{matrix}1-\rho\\ 0\end{matrix}\right) \times G_{1,2}^{2,0}\left(\frac{4m\gamma}{\bar{\gamma}}\,\middle|\,\begin{matrix}1/2\\ 0, m\end{matrix}\right) d\gamma\right] \tag{10}$$

where the integrand is expressed in terms of the Meijer's G-functions with the help of [13, eqs. (8.4.2.5) and (8.4.46.7)]. Evaluating the integral (10) with the help of the identity [13, eq. (2.24.1.1)], we obtain $E_0(\rho)$ in closed form as in (7).

B. Proof of Corollary 1

It follows from (5) that the capacity C is given by

$$\begin{aligned} C &= \frac{1}{2}\left[\frac{\partial E_0(\rho)}{\partial \rho}\right]\Bigg|_{\rho=0} \\ &= \frac{1}{2}\int_0^\infty \ln(1+\gamma)\, p_{\gamma_{\text{end}}}(\gamma)\, d\gamma. \end{aligned} \tag{11}$$

Similar to the derivation of $E_0(\rho)$ in Appendix A, we first express the integrand in terms of the Meijer's G-functions with the help of [13, eqs. (8.4.6.5) and (8.4.46.7)]. Next, using [13, eq. (2.24.1.1)], we obtain (8) and complete the proof.

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