

# Amplify-and-Forward Two-Way Relay Networks: Error Exponents and Resource Allocation

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**Abstract**—In a two-way relay network, two terminals exchange information over a shared wireless half-duplex channel with the help of a relay. Due to its fundamental and practical importance, there has been an increasing interest in this channel. However, there has been little work that characterizes the fundamental tradeoff between the communication reliability and transmission rate across all signal-to-noise ratios. In this paper, we consider amplify-and-forward (AF) two-way relaying due to its simplicity. We first derive the random coding error exponent for the link in each direction. From the exponent expression, the capacity and cutoff rate for each link are also deduced. We then put forth the notion of *bottleneck* error exponent, which is the worst exponent decay between the two links, to give us insight into the fundamental tradeoff between the rate pair and information-exchange reliability in the two-way relay network. As applications of the error exponent analysis to design a reliable AF two-way relay network, we present two optimization framework to maximize the bottleneck error exponent, namely: i) the optimal rate allocation under a sum-rate constraint and its closed-form *quasi-optimal* solution that requires only knowledge of the capacity and cutoff rate of each link; and ii) the optimal power allocation under a total power constraint and perfect global channel state information, which is shown equivalently to a quasi-convex optimization problem. Numerical results verify our analysis and the effectiveness of the optimal rate and power allocations in maximizing the bottleneck error exponent, i.e. the network information-exchange reliability.

**Index Terms**—Amplify-and-forward relaying, bidirectional communication, quasi-convex optimization, random coding error exponent, resource allocation, two-way relay network.

## I. INTRODUCTION

THE two-way communication channel was first introduced by Shannon, showing how to efficiently design message structures to enable simultaneous bidirectional communication at the highest possible data rates [1]. Recently, this model has regained significant interest by introducing an additional relay to support the exchange of information between the

two communicating terminals. The attractive feature of this two-way relay model is that it can compensate the spectral inefficiency of one-way relaying under a half-duplex constraint [2]–[7]. With one-way relaying, we would use four phases to exchange information between two terminals via a half-duplex relay, i.e., it takes two phases to send information from one terminal to the other terminal and two phases for the reverse direction (see Fig. 1). However, exploiting the knowledge of terminals' own transmitted signals and the broadcast nature of the wireless medium, we can improve the spectral efficiency by using only two phases to exchange information in the two-way relay channel (TWRC) [2].

Due to the aforementioned fundamental and practical importance of the TWRC, much work has investigated the sum rate and the achievable rate region of the TWRC with different relaying protocols [2]–[7]. The half-duplex amplify-and-forward (AF) and decode-and-forward (DF) TWRCs have been studied in [2] where it was shown that both protocols with two-way relaying can redeem a significant portion of the half-duplex loss. In [3], the achievable rates for AF, DF, joint-DF, and denoise-and-forward relaying have been analyzed and the conditions for maximization of the two-way rate are investigated for each relaying scheme. The broadcast capacity region in terms of the maximal probability of error has been derived in [5] for the DF TWRC. A new achievable rate region for the TWRC has been found in [6] for partial DF relaying, which is a superposition of both DF and compress-and-forward relaying. Bit error probability at each terminal has also been analyzed for a memoryless additive white Gaussian noise (AWGN) TWRC [7]. However, there has been few work that characterizes the fundamental tradeoff between the communication reliability and transmission rate in the TWRC across all signal-to-noise ratio (SNR) regimes.

In this paper, we consider half-duplex AF two-way relaying due to its simplicity in practical implementation. To characterize the fundamental tradeoff between the communication reliability and rate, we first derive Gallager's random coding error exponent (RCEE)—the classical lower bound to Shannon's reliability function (see, e.g., [8]–[15] and references therein)—for the link of each direction in the AF TWRC.<sup>1</sup> Instead of considering only the achievable rate or error probability as a performance measure, the RCEE results can reveal the inherent tradeoff between these measures to unveil the effectiveness of two-way relaying in redeeming a significant portion of the half-duplex loss in the information exchange. From the exponent expression, the capacity and cutoff rate for

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<sup>1</sup>In the following, we shall use simply the term "TWRC" to denote the AF TWRC.

each link in the TWRC are further deduced. We then introduce the *bottleneck* error exponent, which is defined by the worst exponent decay between the links of two directions, to capture the tradeoff between the rate pair of both links and the reliability of information exchange at such a rate pair. Using this notion, we can appertain a bottleneck exponent value to each rate pair and characterize the bottleneck exponent plane from the set of all possible rate pairs besides the achievable rate region. This enables us to design a two-way relay network with reliable information exchange.

For applications of the error exponent analysis for the design of a reliable two-way network, we present two optimal network resource (rate and power) allocations, the main results of which can be summarized as follows.

- We show that the optimal rate allocation to maximize the bottleneck error exponent under a sum-rate constraint is a rate pair such that the RCEE values of both links become identical at the respective rates. This optimal rate pair can be determined by a closed-form solution for sum rates less than a certain constant—called the *decisive sum rate*—depending only on the cutoff and critical rates of each link. Furthermore, the optimal solution requires only the knowledge of each cutoff rate. At sum rates larger than the decisive point, we can allocate a rate pair *quasi-optimally* in closed form, requiring only knowledge of the capacity and cutoff rate of each link.
- We determine the optimal power allocation that maximizes the bottleneck error exponent under a total power constraint of the two terminals. In the presence of perfect global channel state information (CSI), we show that this power allocation problem can be formulated as a quasi-convex optimization problem, where the optimal solution can be efficiently determined via a sequence of convex feasibility problems in the form of second-order cone programs (SOCPs).

The rest of this paper is organized as follows. In Section II, we describe the system model. In Section III, we present the results of the error exponent analysis for the TWRC. The rate and power optimization framework for two-way relay networks is developed in Section IV. We provide some numerical results in Section V and finally conclude the paper in Section VI.

*Notation:* Throughout the paper, we shall use the following notation. Boldface upper- and lower-case letters denote matrices and column vectors, respectively. The superscript  $(\cdot)^T$  denotes the transpose. We use  $\mathbb{R}$ ,  $\mathbb{R}_+$ , and  $\mathbb{R}_{++}$  to denote the set of real numbers, nonnegative real numbers, and positive real numbers, respectively. We use  $\lceil x \rceil$  to denote the smallest integer greater than or equal to  $x$ . For a random variable  $X$ , we use  $p_X(x)$  to denote the probability density function (PDF) of  $X$ . A circularly symmetric complex Gaussian distribution with mean  $\mu$  and variance  $\sigma^2$  is denoted by  $\mathcal{CN}(\mu, \sigma^2)$  and the exponential distribution with a hazard rate  $\lambda$  is denoted by  $\mathcal{E}(\lambda)$ .

## II. SYSTEM MODEL

We consider the TWRC as illustrated in Fig. 1, where a half-duplex relay node R bidirectionally communicates between

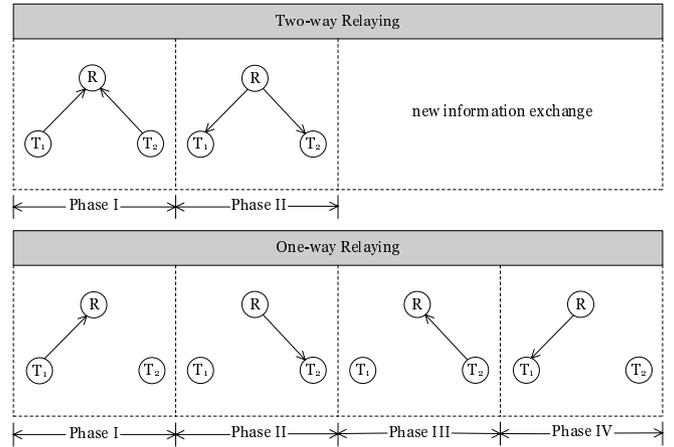


Fig. 1. Information exchange with one- and two-way relaying.

two terminals  $T_k$ ,  $k \in \mathcal{T} = \{1, 2\}$ , with AF relaying. In the multiple-access phase (Phase I), the terminals  $T_k$  transmit their information to the relay and the received signal at the relay is given by

$$Y_R = H_1 X_1 + H_2 X_2 + Z_R \quad (1)$$

where  $X_k$  is the transmitted signal from the terminal  $T_k$  with  $\mathbb{E}\{|X_k|^2\} = p_k$ ,  $H_k \sim \mathcal{CN}(0, \Omega_k)$  is the channel coefficient from  $T_k$  to the relay, and  $Z_R \sim \mathcal{CN}(0, N_0)$  is the complex AWGN.<sup>2</sup> Note that  $|H_k|^2 \sim \mathcal{E}(1/\Omega_k)$ .

At the relay, the received signal is scaled and broadcasted to both terminals in the broadcast phase (Phase II), while satisfying its power constraint  $p_R$ . Then, the received signal at the terminal  $T_k$  is given by

$$Y_k = H_k X_R + Z_k \quad (2)$$

where  $X_R = G Y_R$  is the transmitted signal from the relay with  $\mathbb{E}\{|X_R|^2\} = p_R$ ,  $Z_k \sim \mathcal{CN}(0, N_0)$  is the AWGN, and  $G$  is the relaying gain given by

$$G = \sqrt{\frac{p_R}{p_1 |H_1|^2 + p_2 |H_2|^2 + N_0}}. \quad (3)$$

We impose a total transmit power constraint  $P$  such that  $p_1 + p_2 \leq P$ . As in [2], we further assume that each terminal  $T_k$  knows its own transmitted signal and has perfect CSI to remove self-interference prior to decoding. For notational convenience, we shall refer to the communication link  $T_1 \rightarrow R \rightarrow T_2$  as the link  $L_1$  and  $T_2 \rightarrow R \rightarrow T_1$  as the link  $L_2$ . With the self-interference cancellation and the CSI-assisted AF relaying gain chosen as (3), the effective SNR of the link  $L_k$  is given by

$$\gamma_k^{\text{eff}} = \frac{p_k p_R \alpha_1 \alpha_2}{p_k \alpha_k + (p_R + p_1 p_2 / p_k) \alpha_1 \alpha_2 / \alpha_k + 1} \quad (4)$$

where  $\alpha_k \triangleq |H_k|^2 / N_0$ .

## III. ERROR EXPONENT ANALYSIS

### A. Mathematical Preliminaries

The reliability function or error exponent for a channel of the capacity  $C$  is the best exponent decay with the codeword

<sup>2</sup>We assume the channel reciprocity for  $H_k$  as in [2].

length  $N$  in the average probability of error that one can achieve at a rate  $R < C$  [8]:

$$E(R) \triangleq \limsup_{N \rightarrow \infty} -\frac{1}{N} \ln P_e^{\text{opt}}(R, N) \quad (5)$$

where  $P_e^{\text{opt}}(R, N)$  is the average block error probability for the optimal block code of length  $N$  and rate  $R$ .<sup>3</sup> We now consider the memoryless link  $L_1$  of the TWRC,<sup>4</sup> where the received signal at the terminal  $T_2$  with the cancellation of self interference  $X_2$  is given by

$$\tilde{Y}_2 = H_2 G (H_1 X_1 + Z_R) + Z_2 \quad (6)$$

which is equivalent to a dual-hop AF relay channel [16], [17].

Let  $Q(x_1)$  be an arbitrary probability density assignment on the input  $X_1$  of the terminal  $T_1$  and  $M \geq 2$  be the number of codewords of block length  $N$ . Then, following Gallager's random coding arguments [8, Theorem 5.6.2], the average probability of maximum-likelihood decoding error over the ensemble of  $(N, R)$  block codes for the link  $L_1$  is bounded, for any choice of  $\rho \in [0, 1]$ , by

$$P_e \leq (M - 1)^\rho e^{-N E_0(\rho, Q)} \quad (7)$$

with  $E_0(\rho, Q)$  given by (8), shown at the bottom of the page, where the transition PDF  $p_{\tilde{Y}_2|X_1, H_1, H_2}(\tilde{y}_2|x_1, h_1, h_2)$  is given by

$$p_{\tilde{Y}_2|X_1, H_1, H_2}(\tilde{y}_2|x_1, h_1, h_2) = \frac{1}{\pi N_0 (1 + |h_2 G|^2)} \exp \left\{ -\frac{|\tilde{y}_2 - h_2 G h_1 x_1|^2}{N_0 (1 + |h_2 G|^2)} \right\}. \quad (9)$$

Since the terminals  $T_1$  and  $T_2$  communicate each codeword in  $2N$  symbol-time intervals with the help of the relay, the transmission rate  $R$  in nats/s/Hz is defined as

$$R = \frac{\ln M}{2N} \quad (10)$$

relating the number of codewords to  $R$  as  $M = \lceil e^{2NR} \rceil$ . Hence, using (7) and the fact that  $M - 1 \leq e^{2NR}$ , we obtain

$$P_e \leq \exp[-N \{E_0(\rho, Q) - 2\rho R\}]. \quad (11)$$

In (11), the tightest bound is obtained by choosing the parameter  $\rho \in [0, 1]$  and the input distribution  $Q$  at the rate  $R$  to maximize  $E_0(\rho, Q) - 2\rho R$ . Therefore, Gallager's random coding exponent  $E_r(R)$  for the link  $L_1$  of the TWRC is given

<sup>3</sup>Throughout the paper, we shall use a rate measured in units of nats per second per Hz (nats/s/Hz).

<sup>4</sup>The memoryless property of the channel is considered with perfect interleaving as in [9], [12], [15]. With a stringent delay constraint, we can adopt a block-fading model using a piece-wise approximation of the channel memory [10], [13], [14].

by<sup>5</sup>

$$E_r(R) = \max_{0 \leq \rho \leq 1} \max_Q \{E_0(\rho, Q) - 2\rho R\}. \quad (12)$$

Unfortunately, the double maximization in (12) is generally very difficult since the inner integral is raised to a fractional exponent when  $\rho \in (0, 1)$  and the lack of knowledge about the optimal input distribution  $Q$ . For analytical tractability, the Gaussian input distribution is often used [8]–[15], although this choice of the input distribution gives a lower bound on the optimal random coding exponent and is optimal only if the rate  $R$  approaches the channel capacity. For the link  $L_k$  with the Gaussian input  $X_k \sim \mathcal{CN}(0, p_k)$ , we shall denote Gallager's random coding exponent in (12) and the quantity  $E_0(\rho, Q)$  in (8) by  $E_{r,k}(R)$  and  $E_{0,k}(\rho)$ , respectively.

## B. Two-way Relay Channels

The TWRC consists of two communication links and the achievable rate region can be characterized by the rates of two parallel relay channels under perfect self-interference cancellation [2]. As such, we need to first derive the RCEE for each link and subsequently introduce a notion of the bottleneck error exponent for the TWRC to effectively capture the tradeoff between the individual rates and the reliability. Using the Gaussian input distribution, we obtain the following proposition for the RCEE of each link in the TWRC.

*Proposition 1:* With the Gaussian input distribution, the RCEE for the link  $L_k$  of the TWRC with AF relaying is given by

$$E_{r,k}(R) = \max_{0 \leq \rho \leq 1} \{E_{0,k}(\rho) - 2\rho R\} \quad (13)$$

where

$$E_{0,k}(\rho) = -\ln \mathbb{E}_{\gamma_k^{\text{eff}}} \left\{ \left( 1 + \frac{\gamma_k^{\text{eff}}}{1 + \rho} \right)^{-\rho} \right\}. \quad (14)$$

*Proof:* For the link  $L_1$ , by substituting the Gaussian input  $Q(x_1) = \frac{1}{\pi p_1} e^{-|x_1|^2/p_1}$  into (8) and performing the integrations with respect to  $x_1$  and  $\tilde{y}_2$ , we obtain

$$E_{0,1}(\rho) = -\ln \mathbb{E}_{H_1, H_2} \left\{ \left( 1 + \frac{1}{1 + \rho} \frac{p_1 |H_2 G H_1|^2}{N_0 (1 + |H_2 G|^2)} \right)^{-\rho} \right\}. \quad (15)$$

Since

$$\frac{p_1 |H_2 G H_1|^2}{N_0 (1 + |H_2 G|^2)} = \gamma_1^{\text{eff}} \quad (16)$$

<sup>5</sup>For the link  $L_2$ , we can obtain the RCEE by taking the same steps leading to (12). The variations of [8, Theorem 5.6.2] for single- or multiple-antenna wireless systems have been extensively studied in the literature (see, e.g., [9]–[15]). To develop a tighter random coding bound, Gallager further introduced one more free parameter to be optimized (see [8, Section 7.3] for details). When  $\beta = 1$  (or equivalently,  $r = 0$ ) in [9] and [12] (more generally,  $\beta = n_T$  in [14]), the problem goes back to the maximization problem, as in (12), over only a single free parameter  $\rho \in [0, 1]$ .

$$E_0(\rho, Q) \triangleq -\ln \left\{ \int_{h_2} p_{H_2}(h_2) \int_{h_1} p_{H_1}(h_1) \int_{\tilde{y}_2} \left[ \int_{x_1} Q(x_1) p_{\tilde{Y}_2|X_1, H_1, H_2}(\tilde{y}_2|x_1, h_1, h_2)^{\frac{1}{1+\rho}} dx_1 \right]^{1+\rho} d\tilde{y}_2 dh_1 dh_2 \right\} \quad (8)$$

we complete the proof (with the same steps for the link  $L_2$ ). ■

*Remark 1:* It is difficult to obtain a closed-form solution for (14) in Proposition 1 due to an analytically intractable form of the effective SNR  $\gamma_k^{\text{eff}}$  given in (4). In what follows, to alleviate such difficulty and render (14) more amenable to further analysis, we use the upper bound  $\gamma_k^{\text{ub}}$  on the effective SNR  $\gamma_k^{\text{eff}}$  of CSI-assisted AF relaying by ignoring the term 1 in the denominator:

$$\gamma_k^{\text{ub}} = \frac{p_k p_R \alpha_1 \alpha_2}{p_k \alpha_k + (p_R + p_1 p_2 / p_k) \alpha_1 \alpha_2 / \alpha_k} \quad (17)$$

which corresponds to the *ideal/hypothetical* AF relaying [16]–[18].

*Remark 2:* The factor 2 of  $\rho R$  in (13) is due to the use of two phases for the exchange of information in the TWRC, involving the transmission rate  $R = \frac{\ln M}{2N}$  in (10). In contrast, with one-way relaying, the information exchange occurs over four phases and hence, this factor should be 4, leading the RCEE for each link to  $E_{r,k}(R) = \max_{0 \leq \rho \leq 1} \{E_{0,k}(\rho) - 4\rho R\}$ .<sup>6</sup>

*Theorem 1:* With the Gaussian input distribution, the RCEE for the link  $L_k$  of the TWRC with ideal/hypothetical AF relaying is given by

$$\tilde{E}_{r,k}(R) = \max_{0 \leq \rho \leq 1} \left\{ \tilde{E}_{0,k}(\rho) - 2\rho R \right\} \quad (18)$$

with  $\tilde{E}_{0,k}(\rho)$  given by (19) for  $0 < \rho \leq 1$ , shown at the bottom of the page, and  $\tilde{E}_{0,k}(\rho) = 0$  for  $\rho = 0$ , where  $\Gamma(\cdot)$  is Euler's gamma function,  $H_{E,(A:C),F,(B:D)}^{K,N,N',M,M'}[\cdot]$  is the

<sup>6</sup>We can improve the performance of one-way relaying by exploiting the *cooperative diversity* if the direct links between two terminals  $T_1$  and  $T_2$  are feasible. However, as considered in [2], [16], [17], [19], we assume that these direct links are *infeasible* or have *poor* connection. In the one-way relay channel (OWRC), the relay spends the time period of two phases to exchange two signals  $X_1$  and  $X_2$  between  $T_1$  and  $T_2$  terminals, as shown in Fig. 1. Hence, if the total relaying power for information exchange of these two signals is again constrained to  $p_R$ , then the ideal/hypothetical AF relaying yields the upper bound on the effective SNR for the link  $L_k$  as

$$\gamma_k^{\text{up}} = \frac{p_k (p_R/2) \alpha_1 \alpha_2}{p_k \alpha_k + (p_R/2) \alpha_1 \alpha_2 / \alpha_k}$$

which is slightly different from (17) but makes no deviation in the analysis.

generalized Fox  $H$ -function [20, eq (2.2.1)], and

$$\begin{aligned} \eta_k &= \frac{p_R}{p_R + p_1 p_2 / p_k} \\ \lambda_k &= \frac{N_0}{p_k \Omega_k} \\ \mu_k &= \frac{N_0 \Omega_k}{(p_1 p_2 / p_k + p_R) \Omega_1 \Omega_2}. \end{aligned}$$

*Proof:* See Appendix A. ■

*Remark 3:* The ideal/hypothetical AF relaying yields an upper bound on the error exponent attained by the CSI-assisted AF relaying, while the choice of the Gaussian input gives a lower bound on the *optimal* random coding exponent but still would exhibit better exponent behavior than the input distribution of practical systems. Therefore, these choices can be leveraged to provide a benchmark for practical AF (two-way) relaying. Recently, the random coding exponent for dual-hop (one-way) independent and identically distributed Nakagami- $m$  fading relay channels has been analyzed in [21] with the same arguments.

*Remark 4:* The maximum of the exponent  $\tilde{E}_{r,k}(R)$  over  $\rho$  occurs at  $R = \frac{1}{2} \left[ \frac{\partial \tilde{E}_{0,k}(\rho)}{\partial \rho} \right]_{\rho=\rho_{\text{opt}}}$  and hence, the slope of the exponent–rate curve at a rate  $R$  is equal to  $-2\rho_{\text{opt}}$ . The maximizing  $\rho_{\text{opt}}$  lies in  $[0, 1]$  if

$$R_{\text{cr},k} = \frac{1}{2} \left[ \frac{\partial \tilde{E}_{0,k}(\rho)}{\partial \rho} \right]_{\rho=1} \leq R \leq \frac{1}{2} \left[ \frac{\partial \tilde{E}_{0,k}(\rho)}{\partial \rho} \right]_{\rho=0} = \langle C_k \rangle \quad (20)$$

where  $R_{\text{cr},k}$  and  $\langle C_k \rangle$  are the critical rate and the (ergodic) capacity for the link  $L_k$ , respectively. For  $R < R_{\text{cr},k}$ , we have  $\rho_{\text{opt}} = 1$ , yielding the slope of the exponent–rate curve is equal to  $-2$  and  $\tilde{E}_{r,k}(R) = \tilde{E}_{0,k}(1) - 2R$ . Furthermore, the cutoff rate for the link  $L_k$  is given by  $R_{0,k} = \tilde{E}_{0,k}(1)/2$ . This quantity is equal to the value of  $R$  at which the exponent becomes zero by setting  $\rho = 1$ . While the capacity determines the maximum achievable rate, the cutoff rate determines the maximum practical transmission rate for possible sequential decoding strategies and indicates both the values of the zero-rate exponent and the rate regime in which the error probability can be made arbitrarily small by increasing the codeword length.

*Corollary 1:* The ergodic capacity for the link  $L_k$  of the TWRC with ideal/hypothetical AF relaying is given by (21),

$$\begin{aligned} \tilde{E}_{0,k}(\rho) &= -\ln \mathbb{E}_{\gamma_k^{\text{ub}}} \left\{ \left( 1 + \frac{\gamma_k^{\text{ub}}}{1 + \rho} \right)^{-\rho} \right\} \\ &= -\ln \left\{ \frac{4\lambda_k \mu_k}{\sqrt{\pi} \Gamma(\rho) (\sqrt{\lambda_k} + \sqrt{\mu_k})^4} H_{1,(1:1),0,(1:2)}^{1,1,1,1,2} \left[ \begin{array}{c} \frac{\eta_k}{(1+\rho)(\sqrt{\lambda_k} + \sqrt{\mu_k})^2} \\ \frac{4\sqrt{\lambda_k \mu_k}}{(\sqrt{\lambda_k} + \sqrt{\mu_k})^2} \end{array} \middle| \begin{array}{c} (2, 1) \\ (1 - \rho, 1); (1/2, 1) \\ (0, 1); (0, 1), (0, 1) \end{array} \right] \right. \\ &\quad \left. - \frac{2(\lambda_k + \mu_k) \sqrt{\lambda_k \mu_k}}{\sqrt{\pi} \Gamma(\rho) (\sqrt{\lambda_k} + \sqrt{\mu_k})^4} H_{1,(1:1),0,(1:2)}^{1,1,1,1,2} \left[ \begin{array}{c} \frac{\eta_k}{(1+\rho)(\sqrt{\lambda_k} + \sqrt{\mu_k})^2} \\ \frac{4\sqrt{\lambda_k \mu_k}}{(\sqrt{\lambda_k} + \sqrt{\mu_k})^2} \end{array} \middle| \begin{array}{c} (2, 1) \\ (1 - \rho, 1); (1/2, 1) \\ (0, 1); (1, 1), (-1, 1) \end{array} \right] \right\} \quad (19) \end{aligned}$$

shown at the bottom of the page.

*Proof:* See Appendix B. ■

*Corollary 2:* The cutoff rate for the link  $L_k$  of the TWRC with ideal/hypothetical AF relaying is given by (22), shown at the bottom of the page.

*Proof:* It follows immediately from (19) by setting  $\rho = 1$ . ■

*Remark 5:* It is insufficient to characterize the information exchange in the TWRC by only investigating the RCEE for each link individually, as it just reflects the tradeoff between the communication rate and reliability for the information transmission in one direction. Therefore, we introduce a notion of the *bottleneck exponent* for the TWRC to capture the tradeoff between the rate pair of both links and the reliability of information exchange at such a rate pair, enabling us to optimize the resource allocation in the TWRC.

### C. Bottleneck Error Exponent

*Definition 1 (Bottleneck Error Probability):* For a TWRC with the terminal  $T_k$  transmitting a code  $(\lceil e^{2NR_k} \rceil, N)$  of rate  $R_k$ , the bottleneck error probability is defined as

$$P_e^* \triangleq \max_{k \in \mathcal{T}} P_e^{(k)} \quad (23)$$

where  $P_e^{(k)}$  denotes the error probability of the link  $L_k$ .

Note that Definition 1 can be applicable for a general TWRC, regardless of relaying protocols, motivated by the fact that if the bottleneck error probability  $P_e^*$ , i.e.,  $\max_{k \in \mathcal{T}} P_e^{(k)}$  decreases exponentially with the codeword length  $N$  at a rate pair  $(R_1, R_2)$ , then  $P_e^{(1)} \rightarrow 0$  and  $P_e^{(2)} \rightarrow 0$  (consequently  $P_e^{(1)} + P_e^{(2)} \rightarrow 0$ ) as  $N \rightarrow \infty$ . From the random coding bound  $P_e^{(k)} \leq e^{-N\tilde{E}_{r,k}(R_k)}$ , the bottleneck error probability of the TWRC is bounded by

$$P_e^* \leq \max_{k \in \mathcal{T}} e^{-N\tilde{E}_{r,k}(R_k)}. \quad (24)$$

Using (24), we define the bottleneck error exponent of the TWRC as follows.

*Definition 2 (Bottleneck Error Exponent):* For a TWRC with the terminal  $T_k$  transmitting a code  $(\lceil e^{2NR_k} \rceil, N)$  of rate  $R_k$ , the bottleneck error exponent at the information-exchange rate pair  $(R_1, R_2)$  is defined as

$$E_r^*(R_1, R_2) \triangleq \min_{k \in \mathcal{T}} \tilde{E}_{r,k}(R_k). \quad (25)$$

*Remark 6:* Using the RCEE of the link  $L_k$  in Theorem 1, we can readily obtain the bottleneck error exponent  $E_r^*(R_1, R_2)$ . From Definition 2, we can see that the bottleneck error exponent captures the behavior of the worst exponent decay between the two links in the TWRC and reflects the reliability of the information exchange at a rate pair  $(R_1, R_2)$ . When the worst link is good enough, it means that the other link must also be good. As a result, using (25) as an information-exchange reliability measure, we can design a two-way relay network such that both links can communicate reliably. Besides the achievable rate region, we can also characterize the bottleneck exponent plane from the set of all possible rate pairs. This plane could provide us with further understanding of the tradeoff between the rate pair  $(R_1, R_2)$  and the bottleneck error exponent (i.e., information-exchange reliability).

## IV. OPTIMAL NETWORK RESOURCE ALLOCATION

In this section, we consider the optimal rate and power allocation problems to design a reliable two-way relay network such that the network information-exchange reliability (i.e., the bottleneck error exponent) is maximized.

### A. Optimal Rate Allocation

In the following, we present the optimal rate allocation that maximizes the bottleneck error exponent  $E_r^*(R_1, R_2)$  under

$$\langle C_k \rangle = \frac{2\lambda_k\mu_k}{\sqrt{\pi}(\sqrt{\lambda_k} + \sqrt{\mu_k})^4} H_{1,(2:1),0,(2:2)}^{1,2,1,1,2} \left[ \begin{array}{c} \frac{\eta_k}{(\sqrt{\lambda_k} + \sqrt{\mu_k})^2} \\ \frac{4\sqrt{\lambda_k\mu_k}}{(\sqrt{\lambda_k} + \sqrt{\mu_k})^2} \end{array} \middle| \begin{array}{c} (2, 1) \\ (1, 1), (1, 1); (1/2, 1) \\ \hline (1, 1), (0, 1); (0, 1), (0, 1) \end{array} \right] \\ - \frac{(\lambda_k + \mu_k)\sqrt{\lambda_k\mu_k}}{\sqrt{\pi}(\sqrt{\lambda_k} + \sqrt{\mu_k})^4} H_{1,(2:1),0,(2:2)}^{1,2,1,1,2} \left[ \begin{array}{c} \frac{\eta_k}{(\sqrt{\lambda_k} + \sqrt{\mu_k})^2} \\ \frac{4\sqrt{\lambda_k\mu_k}}{(\sqrt{\lambda_k} + \sqrt{\mu_k})^2} \end{array} \middle| \begin{array}{c} (2, 1) \\ (1, 1), (1, 1); (1/2, 1) \\ \hline (1, 1), (0, 1); (1, 1), (-1, 1) \end{array} \right]. \quad (21)$$

$$R_{0,k} = -\frac{1}{2} \ln \left\{ \frac{4\lambda_k\mu_k}{\sqrt{\pi}(\sqrt{\lambda_k} + \sqrt{\mu_k})^4} H_{1,(1:1),0,(1:2)}^{1,1,1,1,2} \left[ \begin{array}{c} \frac{\eta_k}{2(\sqrt{\lambda_k} + \sqrt{\mu_k})^2} \\ \frac{4\sqrt{\lambda_k\mu_k}}{(\sqrt{\lambda_k} + \sqrt{\mu_k})^2} \end{array} \middle| \begin{array}{c} (2, 1) \\ (0, 1); (1/2, 1) \\ \hline (0, 1); (0, 1), (0, 1) \end{array} \right] \right. \\ \left. - \frac{2(\lambda_k + \mu_k)\sqrt{\lambda_k\mu_k}}{\sqrt{\pi}(\sqrt{\lambda_k} + \sqrt{\mu_k})^4} H_{1,(1:1),0,(1:2)}^{1,1,1,1,2} \left[ \begin{array}{c} \frac{\eta_k}{2(\sqrt{\lambda_k} + \sqrt{\mu_k})^2} \\ \frac{4\sqrt{\lambda_k\mu_k}}{(\sqrt{\lambda_k} + \sqrt{\mu_k})^2} \end{array} \middle| \begin{array}{c} (2, 1) \\ (0, 1); (1/2, 1) \\ \hline (0, 1); (1, 1), (-1, 1) \end{array} \right] \right\}. \quad (22)$$



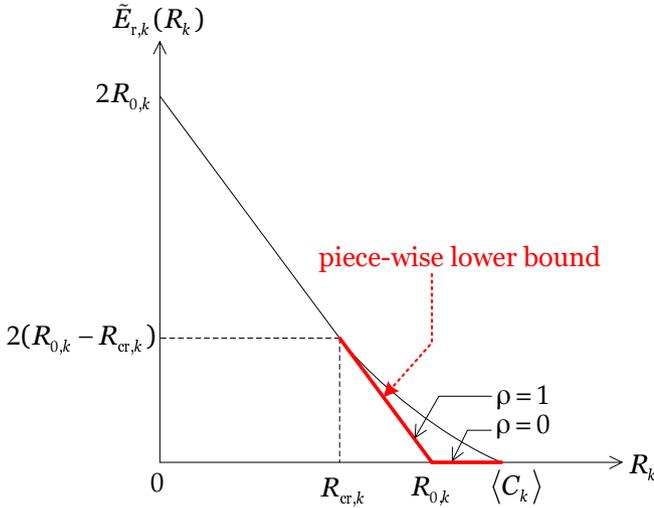


Fig. 3. A piece-wise lower bound to  $\tilde{E}_{r,k}(R_k)$  for  $R_k > R_{cr,k}$ .

rate pair  $(R_1, R_2)$ . Since the error exponent is an information-theoretic quantity as a function of a transmission rate, it is obtained by averaging over all fading states at a given rate for flat fading channels [9]–[14] (see also Proposition 1). Therefore, we can formulate the optimization problem for the rate allocation as in (26). In contrast, for the optimal power allocation that maximizes the error exponent, the transmitter requires CSI knowledge to allocate a transmission power adaptively according to each fading state (see, e.g., [15]).

In the presence of perfect global CSI, for fixed  $\rho$  and  $(R_1, R_2)$ , we are finding the optimal power allocation of  $\mathbf{p}(\alpha_1, \alpha_2) = [p_1(\alpha_1, \alpha_2) \ p_2(\alpha_1, \alpha_2)]^T$  for each fading state. In the following, we will simply use  $\mathbf{p}$ ,  $p_1$  and  $p_2$  to denote  $\mathbf{p}(\alpha_1, \alpha_2)$ ,  $p_1(\alpha_1, \alpha_2)$  and  $p_2(\alpha_1, \alpha_2)$ , respectively. Mathematically, we can formulate this optimization problem as follows:

$$\mathcal{P}_2 = \begin{cases} \max_{\mathbf{p}} & E_r^{\text{int}}(\mathbf{p}, \rho, R_1, R_2) \\ \text{s.t.} & p_1 + p_2 \leq P \\ & p_1 \geq 0, p_2 \geq 0 \end{cases} \quad (30)$$

where<sup>8</sup>

$$E_r^{\text{int}}(\mathbf{p}, \rho, R_1, R_2) = \min_{k \in \mathcal{T}} E_{r,k}^{\text{int}}(\mathbf{p}, \rho, R_k) \quad (31)$$

and  $E_{r,k}^{\text{int}}(\mathbf{p}, \rho, R_k)$  is given by (32), shown at the bottom of the page. Note that the power allocation problem  $\mathcal{P}_2$  in (30) is to optimally design a reliable two-way relay network such that the network reliability is maximized, while constraining the total power (consumed on the network) to  $P$  for supporting a rate pair  $(R_1, R_2)$ . The total power constraint is motivated by the fact that the transmission power is an important network

<sup>8</sup>Note that  $E_r^{\text{int}}(\mathbf{p}, \rho, R_1, R_2)$  does not stand for the instantaneous bottleneck error exponent. It is an instantaneous measure used to determine the bottleneck error exponent in (33).

resource that affects both the lifetime and the scalability of the network [19]. For example, in a wireless sensor network where sensor nodes have limited power resources, the two sensor nodes become the two terminals that are helped by a relay node for exchanging information. Moreover, regulatory agencies may limit the total transmission power to reduce network interference. For example, in a cognitive radio network, the two terminals can be secondary users that are exchanging information through a relay node, and naturally the total power constraint is present to limit the amount of interference at the primary users.

With the optimizing  $\mathbf{p}_{\text{opt}}$  obtained by solving the problem (30), we can find the bottleneck error exponent with optimal power allocation as follows [13]:<sup>9</sup>

$$E_r^*(R_1, R_2) = \max_{0 \leq \rho \leq 1} \left\{ -\ln \mathbb{E}_{\alpha_1, \alpha_2} \left\{ \exp \left[ -E_r^{\text{int}}(\mathbf{p}_{\text{opt}}, \rho, R_1, R_2) \right] \right\} \right\}. \quad (33)$$

Since  $p_1$  and  $p_2$  are positive, we can define  $\psi_k \triangleq \sqrt{p_k}$  and  $\boldsymbol{\psi} = [\psi_1 \ \psi_2]^T$  without loss of optimality. With this change of variables, we can transform the optimization problem in (30) into a quasi-concave program.<sup>10</sup>

*Theorem 3:* For fixed  $\rho$  and rate pair  $(R_1, R_2)$ , the function  $E_r^{\text{int}}(\mathbf{p}, \rho, R_1, R_2)$  is not concave but quasi-concave, and the program  $\mathcal{P}_2$  is quasi-concave.

*Proof:* See Appendix D. ■

It is well known that we can solve quasi-convex optimization problems efficiently through a sequence of convex feasibility problems using the bisection method [22].<sup>11</sup> We formalize it in the following corollary.

*Corollary 3:* The program  $\mathcal{P}_2$  can be solved numerically using the bisection method:

0. Initialize  $t_{\min}$  and  $t_{\max}$ , where  $t_{\min}$  and  $t_{\max}$  define a range of relevant values of  $E_r^{\text{int}}(\mathbf{p}, \rho, R_1, R_2)$ , and set the tolerance  $\varepsilon \in \mathbb{R}_{++}$ .
1. Solve the convex feasibility program  $\mathcal{P}_{\text{socp}}(t)$  in (34) by fixing  $t = (t_{\max} + t_{\min})/2$ .
2. If  $\mathcal{S}(t) = \emptyset$ , then set  $t_{\max} = t$  else set  $t_{\min} = t$ .
3. Stop if the gap  $(t_{\max} - t_{\min})$  is less than the tolerance  $\varepsilon$ . Go to Step 1 otherwise.
4. Output  $\boldsymbol{\psi}_{\text{opt}}$  obtained from solving  $\mathcal{P}_{\text{socp}}(t)$  in Step 1.

<sup>9</sup>In general, the optimization over  $\rho$  for each fading state yields a tighter random coding bound, which is however more difficult to evaluate [12], [13]. Therefore, for each  $\rho \in [0, 1]$ , we first find the optimal power allocation for  $p_1$  and  $p_2$  by solving the problem  $\mathcal{P}_2$  in (30), and then optimize  $\rho$  after averaging over fading states.

<sup>10</sup>Let  $\mathcal{S}$  be a convex subset of  $\mathbb{R}^N$ . A function  $f: \mathcal{S} \rightarrow \mathbb{R}$  is said to be quasi-convex if and only if its lower-level sets  $L(f, a) = \{\mathbf{x} \in \mathcal{S} : f(\mathbf{x}) \leq a\}$  are convex sets for every  $a \in \mathbb{R}$ . Similarly,  $f$  is said to be quasi-concave if and only if its upper-level sets  $U(f, a) = \{\mathbf{x} \in \mathcal{S} : f(\mathbf{x}) \geq a\}$  are convex sets for every  $a \in \mathbb{R}$ .

<sup>11</sup>Note that the program  $\mathcal{P}_2$  is always feasible as long as  $P > 0$ .

$$E_{r,k}^{\text{int}}(\mathbf{p}, \rho, R_k) \triangleq -\ln \left[ 1 + \frac{1}{1 + \rho p_k \alpha_k + (p_R + p_1 p_2 / p_k) \alpha_1 \alpha_2 / \alpha_k + 1} \right]^{-\rho} - 2\rho R_k. \quad (32)$$

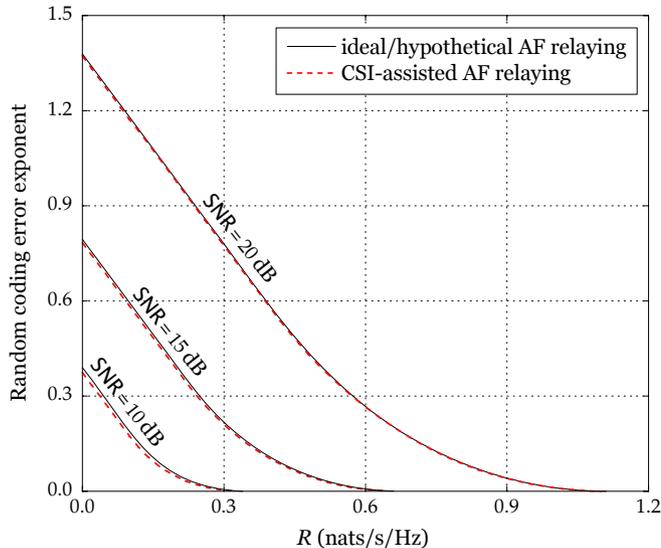


Fig. 4. Random coding error exponent for the link  $L_1$  of the TWRC with ideal/hypothetical AF relaying and CSI-assisted AF relaying.  $\Omega_1 = 0.5$ ,  $\Omega_2 = 2$ , and SNR = 10, 15, and 20 dB.

where the convex feasibility program can be written in SOCP form as [23]

$$\begin{aligned} \mathcal{P}_{\text{socp}}(t) : & \text{find } \boldsymbol{\psi} \\ & \text{s.t. } \boldsymbol{\psi} \in \mathcal{S}(t) \end{aligned} \quad (34)$$

with the set  $\mathcal{S}(t)$  given by

$$\mathcal{S}(t) = \left\{ \boldsymbol{\psi} \in \mathbb{R}_+^2 : \begin{aligned} & \begin{bmatrix} \boldsymbol{\psi}^T \mathbf{e}_1 / \sqrt{v_1} \\ \frac{\mathbf{A}\boldsymbol{\psi}}{\sqrt{1 + p_R \alpha_2}} \end{bmatrix} \succeq_{\mathcal{K}} 0, \\ & \begin{bmatrix} \boldsymbol{\psi}^T \mathbf{e}_2 / \sqrt{v_2} \\ \frac{\mathbf{A}\boldsymbol{\psi}}{\sqrt{1 + p_R \alpha_1}} \end{bmatrix} \succeq_{\mathcal{K}} 0, \\ & \begin{bmatrix} \sqrt{P} \\ \boldsymbol{\psi} \end{bmatrix} \succeq_{\mathcal{K}} 0 \end{aligned} \right\} \quad (35)$$

where  $\succeq_{\mathcal{K}}$  denotes the generalized inequality with respect to the second-order cone (SOC)  $\mathcal{K}$  [22],  $\mathbf{e}_k$  is a standard basis vector with a one at the  $k$ th element,  $\mathbf{A}$  and  $v_k$  are defined by (58) and (59) in Appendix D, respectively.

*Proof:* It follows directly from the proof of Theorem 3 and [23] that we can represent the convex constraints in the set  $\mathcal{S}(t)$  in terms of SOC constraints. ■

*Remark 8:* It is important that we initialize an interval that contains the optimal solution. In our case, we can always let  $t_{\min}$  correspond to the uniform power allocation and we only need to choose  $t_{\max}$  such that the convex feasibility program becomes infeasible.

## V. NUMERICAL RESULTS

In this section, we provide some numerical results to illustrate our analysis. In all examples, we choose  $\Omega_1 = 0.5$ ,  $\Omega_2 = 2$ ,  $p_R = P$ , and define  $\text{SNR} \triangleq P/N_0$ . Without power allocation, we further consider equal power allocation between two terminals  $T_1$  and  $T_2$ , namely,  $p_1 = p_2 = P/2$ .<sup>12</sup>

<sup>12</sup>For one-way relaying, the RCEE, capacity, and cutoff rate are symmetric and equal for two links in the case of equal power allocation.

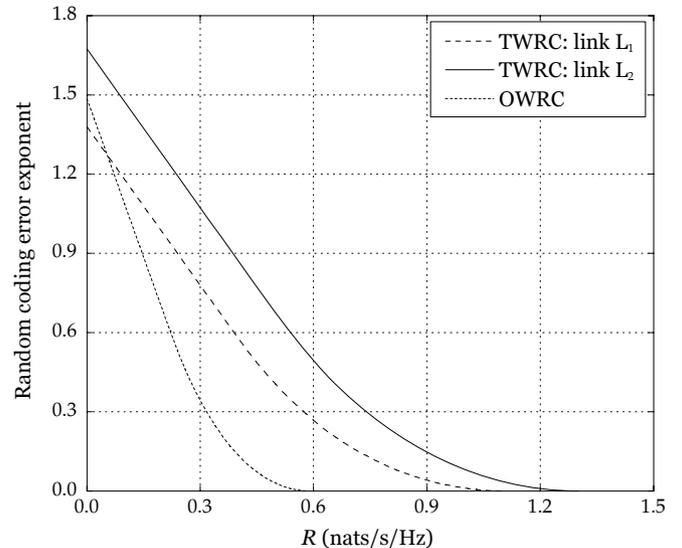


Fig. 5. Random coding error exponent for the link  $L_k$  of the TWRC and OWRC with ideal/hypothetical AF relaying.  $\Omega_1 = 0.5$ ,  $\Omega_2 = 2$ , and SNR = 20 dB.

### A. Random Coding Error Exponent

We first compare the error exponents attained by ideal/hypothetical AF relaying, which yields the upper-bounding SNR  $\gamma_k^{\text{ub}}$  in (17), and the CSI-assisted AF relaying that leads to the SNR  $\gamma_k^{\text{eff}}$  in (4). Fig. 4 shows the RCEE for the link  $L_1$  of the TWRC with ideal/hypothetical AF relaying and CSI-assisted AF relaying at SNR = 10, 15, and 20 dB. We can see that the ideal/hypothetical AF relaying gives a remarkably tight bound—particularly at high SNR—to the error exponent achieved by the CSI-assisted AF relaying. Thus, as reported in [16], it can serve to provide a benchmark for all practical AF relaying with the source-to-relay CSI.

To ascertain the effectiveness of two-way relaying in terms of the error exponent, Fig. 5 shows the RCEE for the link  $L_k$  of the TWRC and OWRC with ideal/hypothetical AF relaying at SNR = 20 dB. To calculate the RCEE, we use Theorem 1 for two-way relaying, whereas we modify Theorem 1 for one-way relaying in such a manner as described in Remark 2. We can see from the figure that the link  $L_2$  of the TWRC shows better exponent behavior than the link  $L_1$  at every rate  $R$  due to the fact that  $\Omega_2 > \Omega_1$ . In the regime below the critical rate, the exponent of the TWRC decreases with the rate twice as slow as in the OWRC and hence, we require to increase the codeword length slowly with two-way relaying to achieve the same level of reliable information exchange as the rate increases. This is due to the spectral efficiency of two-way relaying that requires only half the time duration of one-way relaying to exchange the information. For example, at  $R = 0.3$  (nats/s/Hz), the error exponent is equal to 0.34 for the OWRC, while 0.78 and 1.07 for the links  $L_1$  and  $L_2$  in the TWRC, respectively. These values indicate that to achieve the same communication reliability at  $R = 0.3$ , the required codeword lengths for the links  $L_1$  and  $L_2$  of two-way relaying can be reduced respectively to almost 50% and 33% of the length for one-way relaying.

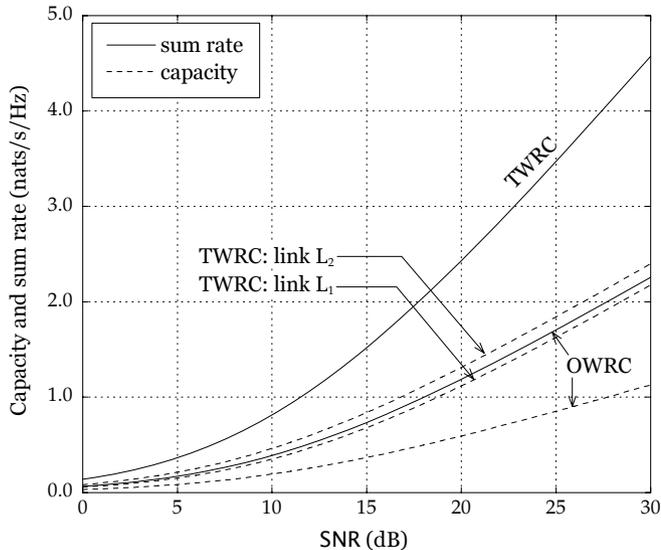


Fig. 6. Capacity and achievable sum rate versus SNR for the link  $L_k$  of the TWRC and OWRC with ideal/hypothetical AF relaying.  $\Omega_1 = 0.5$  and  $\Omega_2 = 2$ .

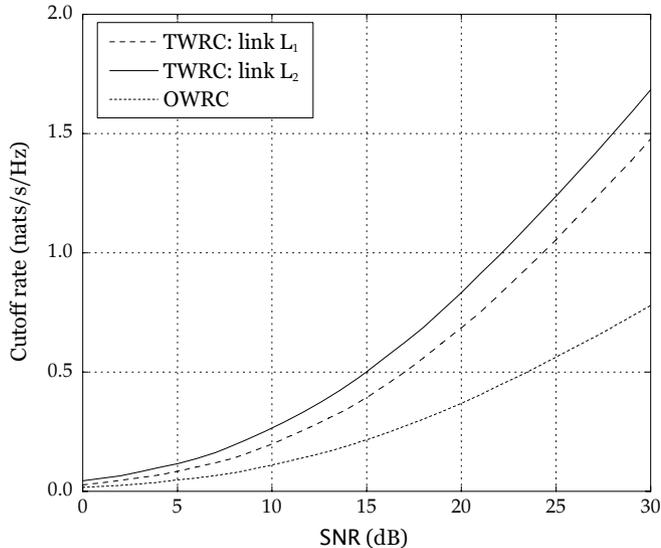


Fig. 7. Cutoff rate versus SNR for the link  $L_k$  of the TWRC and OWRC with ideal/hypothetical AF relaying.  $\Omega_1 = 0.5$  and  $\Omega_2 = 2$ .

### B. Capacity and Cutoff Rate

Figs. 6 and 7 demonstrate the effectiveness of two-way relaying on the achievable rates, where the capacity (or achievable sum rate) and cutoff rate versus SNR are depicted for the link  $L_k$  of the TWRC and OWRC with ideal/hypothetical AF relaying, respectively. We can see from the figures that the slopes of the capacity, achievable sum rate, and cutoff rate curves at high SNR are twice as large in the TWRC as in the OWRC due to again the fact that two-way relaying for the information exchange can reduce the spectral efficiency loss of half-duplex signaling by half in the TWRC. Hence, as can be seen in Fig. 6, the high-SNR slope of the capacity for the link  $L_k$  of the TWRC is identical to that of the achievable sum rate in the OWRC.

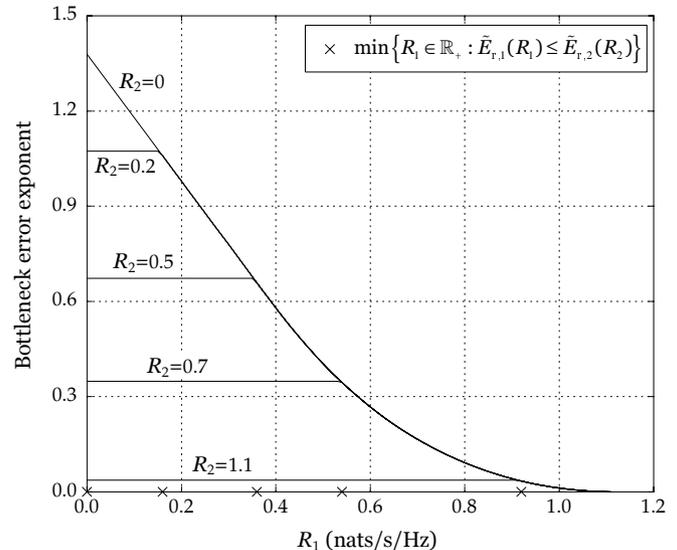


Fig. 8. Bottleneck error exponent  $E_r^*(R_1, R_2)$  versus  $R_1$  for the TWRC with ideal/hypothetical AF relaying at  $R_2 = 0, 0.2, 0.5, 0.7,$  and  $1.1$  nats/s/Hz.  $\Omega_1 = 0.5$ ,  $\Omega_2 = 2$ , and SNR = 20 dB. The values of  $\min\{R_1 \in \mathbb{R}_+ : \tilde{E}_{r,1}(R_1) \leq \tilde{E}_{r,2}(R_2)\}$  are equal to 0, 0.16, 0.36, 0.54, 0.92 nats/s/Hz for  $R_2 = 0, 0.2, 0.5, 0.7,$  and  $1.1$  nats/s/Hz, respectively (indicated by the cross marks).

### C. Bottleneck Error Exponent

To demonstrate the tradeoff between the rate pair and information-exchange reliability in the TWRC, Fig. 8 shows the bottleneck error exponent  $E_r^*(R_1, R_2)$  versus  $R_1$  for the TWRC with ideal/hypothetical AF relaying at SNR = 20 dB when  $R_2 = 0, 0.2, 0.5, 0.7,$  and  $1.1$ . For fixed  $R_2$ , the bottleneck error exponent  $E_r^*(R_1, R_2)$  as a function of  $R_1$  behaves identically to  $\tilde{E}_{r,1}(R_1)$  at  $R_1 \geq R_1^{\min} = \min\{R_1 \in \mathbb{R}_+ : \tilde{E}_{r,1}(R_1) \leq \tilde{E}_{r,2}(R_2)\}$ , whereas  $E_r^*(R_1, R_2)$  is limited to  $\tilde{E}_{r,1}(R_1^{\min})$  for all  $R_1 \leq R_1^{\min}$ . In this example, the values of  $R_1^{\min}$  are equal to 0, 0.16, 0.36, 0.54, 0.92 for  $R_2 = 0, 0.2, 0.5, 0.7,$  and  $1.1$ , respectively. As can be seen, the bad link in terms of the exponent is a bottleneck that limits reliable information exchange, and the bottleneck exponent at large  $R_1$  or  $R_2$  becomes small, indicating the achievable reliability of information exchange would be low at such rate pairs. For example, when  $R_2 = 0.5$  and  $R_1 < 0.36$ , the bottleneck exponent does not increase even  $R_1$  decreases. This is due to the fact that the link  $L_2$  acts as a bottleneck in this rate-pair regime.

### D. Optimal Network Resource Allocation

We now give application examples of the error exponent analysis for the resource allocation in the TWRC. The error exponent serves to indicate a coding requirement for a certain level of communication reliability and hence, it can be used to estimate the codeword length required to achieve a prescribed error probability (see, e.g., [12], [14] for the connection between the error exponent and coding complexity). Therefore, optimally allocating network resources to maximize the bottleneck exponent, we can design a reliable two-way relay network being capable of: i) minimizing the required coding complexity for a prescribed network information-exchange

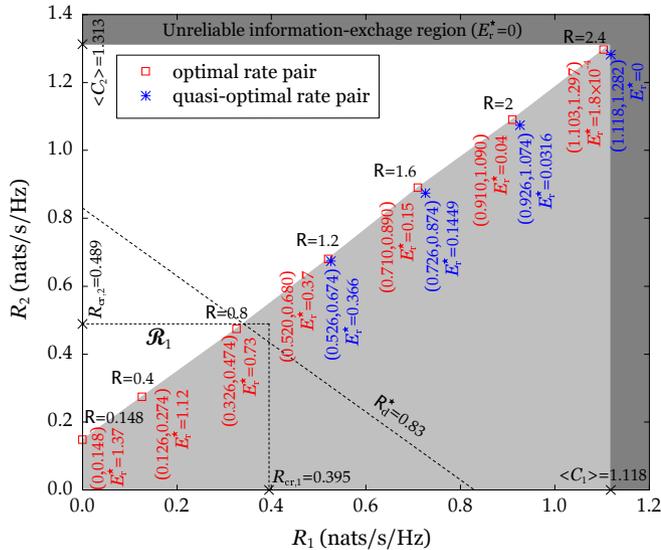


Fig. 9. Optimal rate pair  $(R_1, R_2)_{\text{opt}}$  that maximizes the bottleneck error exponent  $E_r^*(R_1, R_2)$  for the TWRC with ideal/hypothetical AF relaying at sum rates  $R = 0.148, 0.4, 0.8, 1.2, 1.6, 2.0,$  and  $2.4$  nats/s/Hz.  $\Omega_1 = 0.5, \Omega_2 = 2,$  and  $\text{SNR} = 20$  dB. For  $R > R_d^* = 0.83,$  the quasi-optimal rate pairs are also plotted for  $R = 1.2, 1.6, 2.0,$  and  $2.4$  nats/s/Hz.

reliability or ii) maximizing the network reliability with fixed coding complexity.

1) *Optimal Rate Allocation:* Fig. 9 shows the optimal rate pair  $(R_1, R_2)_{\text{opt}}$  that maximizes the bottleneck error exponent  $E_r^*(R_1, R_2)$  under a sum rate constraint for the TWRC with ideal/hypothetical AF relaying at  $\text{SNR} = 20$  dB. The quasi-optimal rate pairs are also plotted for the sum rate  $R > R_d^*$ . The optimal and quasi-optimal rate pairs are determined using Theorem 2 and (29), respectively. The optimal rate pairs  $(R_1, R_2)_{\text{opt}}$  at the sum rates  $R = 0.148, 0.4, 0.8, 1.2, 1.6, 2.0,$  and  $2.4$  are  $(0, 0.148), (0.126, 0.274), (0.326, 0.474), (0.520, 0.680), (0.710, 0.890), (0.910, 1.090),$  and  $(1.103, 1.297),$  attaining the maximum  $E_r^*(R_1, R_2)$  equal to  $1.37, 1.12, 0.73, 0.37, 0.15, 0.04,$  and  $1.8 \times 10^{-4},$  respectively. For  $R > R_d^* = 0.83,$  the quasi-optimal rate pairs at the sum rates  $R = 1.2, 1.6, 2.0,$  and  $2.4$  are  $(0.526, 0.674), (0.726, 0.874), (0.926, 1.074),$  and  $(1.118, 1.282),$  attaining  $E_r^*(R_1, R_2)$  equal to  $0.366, 0.1449, 0.0316,$  and  $0,$  respectively. We can see that the quasi-optimal rate pairs quite well approximate the optimal  $(R_1, R_2)_{\text{opt}}$  for  $R > R_d^*$  and achieve the bottleneck exponents very close to the maximum achievable  $E_r^*(R_1, R_2)$  at such sum rates. In the figure, the region  $\mathcal{R}$  can be divided by the optimal rate curve into two rate-pair subregions in which each RCEE  $\tilde{E}_{r,k}(R_k)$  is dominant for the bottleneck exponent  $E_r^*(R_1, R_2)$ : for example,  $\tilde{E}_{r,1}(R_1)$  is dominant in the light gray subregion, i.e.,  $E_r^*(R_1, R_2) = \tilde{E}_{r,1}(R_1)$ .

The effectiveness of the optimal/quasi-optimal rate allocation in maximizing the bottleneck error exponent can be further ascertained by referring Fig. 10, where the bottleneck error exponent  $E_r^*(R_1, R_2)$  versus  $R_1$  is depicted for the TWRC with ideal/hypothetical AF relaying at  $\text{SNR} = 20$  dB when the sum rate  $R = R_1 + R_2$  is fixed to  $0.5, 1,$  and  $1.5,$  respectively. As can be seen from the figure, the bottleneck exponent  $E_r^*(R_1, R_2)$  is unimodal as a function

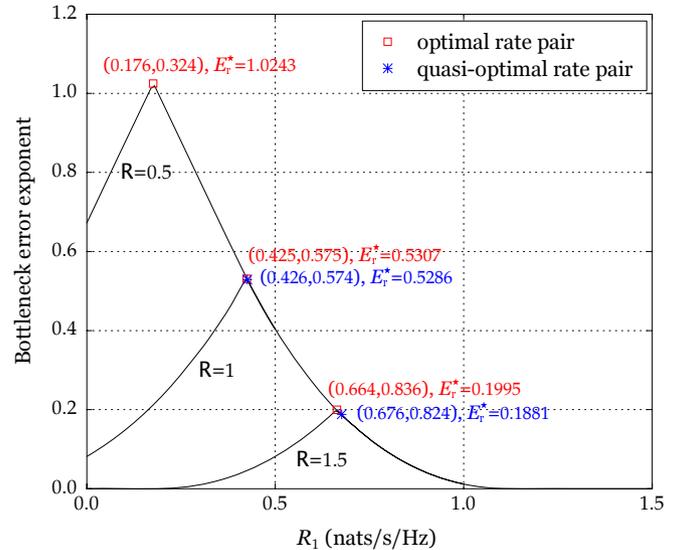


Fig. 10. Bottleneck error exponent  $E_r^*(R_1, R_2)$  versus  $R_1$  for the TWRC with ideal/hypothetical AF relaying at sum rates  $R = 0.5, 1,$  and  $1.5$  nats/s/Hz.  $\Omega_1 = 0.5, \Omega_2 = 2,$  and  $\text{SNR} = 20$  dB. The optimal rate pairs  $(R_1, R_2)_{\text{opt}}$  for each sum rate and the quasi-optimal rate pairs for  $R > R_d^* = 0.83$  are also plotted.

of  $R_1$  for fixed  $R,$  and its maximum is at the mode of  $R_1$  determined by Theorem 2 for each value of  $R.$  We can also observe that the optimal/quasi-optimal rate allocation is of significant benefit to the bottleneck exponent. The optimal rate pairs  $(R_1, R_2)_{\text{opt}}$  at the sum rates  $R = 0.5, 1,$  and  $1.5$  are  $(0.176, 0.324), (0.425, 0.575),$  and  $(0.664, 0.836),$  attaining the maximum  $E_r^*(R_1, R_2)$  equal to  $1.0243, 0.5307,$  and  $0.1995,$  respectively. For  $R > R_d^* = 0.83,$  the quasi-optimal rate pairs at the sum rates  $R = 1$  and  $1.5$  are  $(0.426, 0.574)$  and  $(0.676, 0.824),$  attaining  $E_r^*(R_1, R_2)$  equal to  $0.5286$  and  $0.1881,$  respectively. We can see again that the quasi-optimal rate pairs quite well approximate the optimal ones for  $R > R_d^*$  with a negligible loss in the bottleneck exponent.

2) *Optimal Power Allocation:* Fig. 11 shows the bottleneck error exponent  $E_r^*(R_1, R_2)$  versus  $R = R_1 = R_2$  for the TWRC with ideal/hypothetical AF relaying under optimal and uniform power allocations at  $\text{SNR} = 5$  dB,  $20$  dB, and  $30$  dB. To determine  $E_r^*(R_1, R_2)$  in (33), we first find the optimal power allocation  $\mathbf{p}_{\text{opt}}$  for each  $\rho \in [0, 1]$  using Corollary 3, then successively perform the expectation  $\mathbb{E}_{\alpha_1, \alpha_2} \{ \exp[-E_r^{\text{int}}(\mathbf{p}_{\text{opt}}, \rho, R_1, R_2)] \}$  with respect to  $\alpha_1$  and  $\alpha_2$  by the Monte Carlo method, and maximize  $-\ln \mathbb{E}_{\alpha_1, \alpha_2} \{ \exp[-E_r^{\text{int}}(\mathbf{p}_{\text{opt}}, \rho, R_1, R_2)] \}$  over  $\rho \in [0, 1]$  using the method given in [11, Section 2.2.4]. Compared with the uniform power allocation, we can see that the optimal power allocation improves the bottleneck error exponent in different SNR regimes. As  $R$  is approaching zero, the bottleneck error exponents  $E_r^*(R_1, R_2)$  at  $\text{SNR} = 5$  dB,  $20$  dB, and  $30$  dB become  $0.1424, 1.3793,$  and  $2.9450,$  respectively, for the uniform power allocation, while  $0.1970, 1.5284,$  and  $3.1323$  for the optimal power allocation, respectively. Hence, the bottleneck error exponent can be improved to almost  $138.3\%, 110.8\%,$  and  $106.4\%$  at  $\text{SNR} = 5$  dB,  $20$  dB, and  $30$  dB, respectively, by optimally designing the power allocation, as  $R$  is approaching zero. This reveals that the

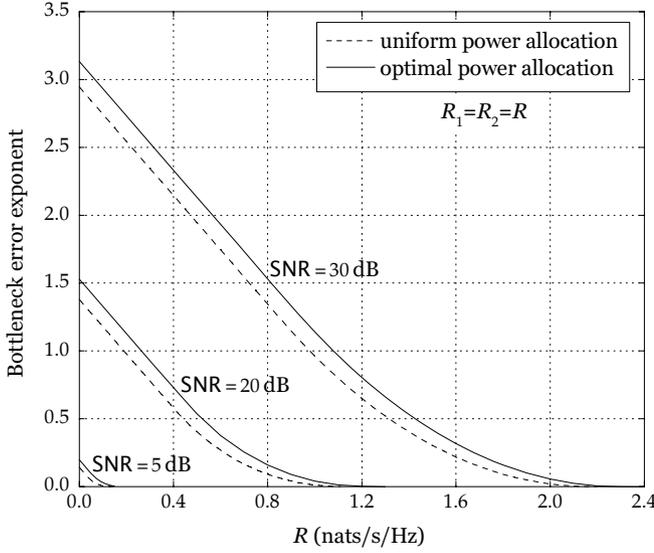


Fig. 11. Bottleneck error exponent  $E_r^*(R_1, R_2)$  versus  $R$  for the TWRC with ideal/hypothetical AF relaying with optimal and uniform power allocations.  $R_1 = R_2 = R$ ,  $\Omega_1 = 0.5$ ,  $\Omega_2 = 2$ , and SNR = 5 dB, 20 dB, and 30 dB.

power allocation design is more effective in the low-SNR regime, whereas the uniform power allocation works relatively well, exhibiting a small portion of exponent degradation, at high SNR.

## VI. CONCLUSIONS

In this paper, we have derived Gallager's random coding exponent to analyze the fundamental tradeoff between the communication reliability and transmission rate in AF two-way relay channels. The exponent has been expressed in terms of the generalized Fox  $H$ -function, from which the capacity and cutoff rate were also deduced for the link of each direction in the TWRC. Using the worst exponent decay between two links as the reliability measure for the information exchange, we put forth the concept of bottleneck error exponent to effectively capture the tradeoff between the rate pair and the information-exchange reliability such that both links can communicate reliably. As its applications, we formulated the optimal rate and power allocation problems that maximize the bottleneck error exponent. Specifically, we presented the optimal rate allocation under a sum-rate constraint and its simple closed-form quasi-optimal solution that requires knowing only the capacity and cutoff rate of each link. The optimal power allocation under a total power constraint of the two terminals was further determined in the presence of perfect global CSI by solving a quasi-convex optimization problem.

## APPENDIX

### A. Proof of Theorem 1

Let  $V_k = p_k \alpha_k$  and  $W_k = (p_R + p_1 p_2 / p_k) \alpha_1 \alpha_2 / \alpha_k$ . Then,  $V_k \sim \mathcal{E}(\lambda_k)$ ,  $W_k \sim \mathcal{E}(\mu_k)$ , and

$$\gamma_{k \in \mathcal{T}}^{\text{ub}} = \frac{\eta_k V_k W_k}{V_k + W_k}. \quad (36)$$

Using the PDF of the Harmonic mean of the two exponential random variables [16] and the transformation  $p_Y(y) = \frac{1}{|a|} p_X(y/a)$  where  $Y = aX$ , we obtain the PDF of  $\gamma_k^{\text{ub}}$  as

$$\begin{aligned} p_{\gamma_k^{\text{ub}}}(\gamma) &= \frac{4}{\eta_k^2} \lambda_k \mu_k \gamma e^{-\frac{(\lambda_k + \mu_k)\gamma}{\eta_k}} K_0 \left( \frac{2\gamma \sqrt{\lambda_k \mu_k}}{\eta_k} \right) \\ &\quad + \frac{2}{\eta_k^2} (\lambda_k + \mu_k) \sqrt{\lambda_k \mu_k} \gamma e^{-\frac{(\lambda_k + \mu_k)\gamma}{\eta_k}} K_1 \left( \frac{2\gamma \sqrt{\lambda_k \mu_k}}{\eta_k} \right) \end{aligned} \quad (37)$$

where  $\gamma \geq 0$  and  $K_\nu(\cdot)$  is the  $\nu$ th order modified Bessel function of the second kind whose integral representation is given by [24, eq. (8.432.6)].

Using (37), we have

$$\tilde{E}_{0,k}(\rho) = -\ln \left\{ \int_0^\infty \left( 1 + \frac{\gamma}{1+\rho} \right)^{-\rho} p_{\gamma_k^{\text{ub}}}(\gamma) d\gamma \right\}. \quad (38)$$

Since it is obvious that  $\tilde{E}_{0,k}(\rho) = 0$  for  $\rho = 0$ , we define

$$\mathcal{I}(\rho) \triangleq \int_0^\infty x (1+ax)^{-\rho} e^{-bx} K_\nu(cx) dx \quad (39)$$

to find  $\tilde{E}_{0,k}(\rho)$  in (38) for  $0 < \rho \leq 1$ . To evaluate the integral  $\mathcal{I}(\rho)$ , we first express  $(1+ax)^{-\rho}$  and  $e^{cx} K_\nu(cx)$  in terms of the Fox  $H$ -functions with the help of [25, eqs. (8.3.2.21), (8.4.2.5), and (8.4.23.5)] as follows:

$$(1+ax)^{-\rho} = \frac{1}{\Gamma(\rho)} H_{1,1}^{1,1} \left[ ax \left| \begin{matrix} (1-\rho, 1) \\ (0, 1) \end{matrix} \right. \right] \quad (40)$$

$$e^{cx} K_\nu(cx) = \frac{\cos(\nu\pi)}{\sqrt{\pi}} H_{1,2}^{2,1} \left[ 2cx \left| \begin{matrix} (1/2, 1) \\ (\nu, 1), (-\nu, 1) \end{matrix} \right. \right] \quad (41)$$

where  $H_{p,q}^{m,n}[\cdot]$  is the Fox  $H$ -function [25, eq. (8.3.1.1)]. Then, substituting (40) and (41) into (39), we have (42), as shown at the bottom of this page, where the last equality follows from [20, eq. (2.6.2)]. Finally, from (37)–(39) and (42), we get (19) and complete the proof.

$$\begin{aligned} \mathcal{I}(\rho) &= \frac{\cos(\nu\pi)}{\sqrt{\pi}\Gamma(\rho)} \int_0^\infty x e^{-(b+c)x} H_{1,1}^{1,1} \left[ ax \left| \begin{matrix} (1-\rho, 1) \\ (0, 1) \end{matrix} \right. \right] H_{1,2}^{2,1} \left[ 2cx \left| \begin{matrix} (1/2, 1) \\ (\nu, 1), (-\nu, 1) \end{matrix} \right. \right] dx \\ &= \frac{\cos(\nu\pi)}{\sqrt{\pi}\Gamma(\rho)} (b+c)^{-2} H_{1,(1:1),0,(1:2)}^{1,1,1,1,2} \left[ \begin{matrix} \frac{a}{b+c} \left| \begin{matrix} (2, 1) \\ (1-\rho, 1); (1/2, 1) \end{matrix} \right. \\ \frac{2c}{b+c} \left| \begin{matrix} (0, 1); (\nu, 1), (-\nu, 1) \end{matrix} \right. \end{matrix} \right]. \end{aligned} \quad (42)$$

### B. Proof of Corollary 1

It follows from Theorem 1 that

$$\langle C_k \rangle = \frac{1}{2} \left[ \frac{\partial \tilde{E}_{0,k}(\rho)}{\partial \rho} \right] \Big|_{\rho=0} = \frac{1}{2} \int_0^\infty \ln(1+\gamma) p_{\gamma_k^{\text{ub}}}(\gamma) d\gamma. \quad (43)$$

Similar to the derivation of  $\tilde{E}_{0,k}(\rho)$ , we first express  $\ln(1+\gamma)$  in terms of the Fox  $H$ -function with the help of [25, eq. (8.4.6.5)] as

$$\ln(1+\gamma) = H_{2,2}^{1,2} \left[ \gamma \left| \begin{matrix} (1,1), (1,1) \\ (1,1), (0,1) \end{matrix} \right. \right]. \quad (44)$$

Then, again using (41) and [20, eq. (2.6.2)], we evaluate (43) as (21) and complete the proof.

### C. Proof of Theorem 2

Since the exponent  $\tilde{E}_{r,k}(R_k)$  is a monotonically decreasing function in  $R_k$ , it is obvious that for any rate pair  $(\dot{R}_1, \dot{R}_2) \in \mathcal{R}$  with the sum rate  $\dot{R}_1 + \dot{R}_2 = R \geq |R_{0,1} - R_{0,2}|$ ,

$$E_r^*(R_1, R_2) \geq E_r^*(\dot{R}_1, \dot{R}_2) \quad (45)$$

whenever  $(R_1, R_2)$  is such that  $\tilde{E}_{r,1}(R_1) = \tilde{E}_{r,2}(R_2)$  and  $R_1 + R_2 = R$ . Therefore, the optimal solution  $(R_1, R_2)_{\text{opt}}$  of the problem  $\mathcal{P}_1$  for  $R \geq |R_{0,1} - R_{0,2}|$  is uniquely given by

$$(R_1, R_2)_{\text{opt}} \in \left\{ (R_1, R_2) \in \mathcal{R} : \tilde{E}_{r,1}(R_1) = \tilde{E}_{r,2}(R_2), \right. \\ \left. R_1 + R_2 = R \right\}. \quad (46)$$

Although, clearly, the optimization problem (26) is mathematically challenging, it follows from (46) that the optimal solution  $(R_1, R_2)_{\text{opt}}$  for  $R \geq |R_{0,1} - R_{0,2}|$  is the intersection point of the rate-pair curve  $\mathcal{C}$  and straight line  $\mathcal{L}$ , and we can determine it graphically, as shown in Fig. 2.

Let  $\mathcal{R}_1 = \{(R_1, R_2) \in \mathcal{R} : 0 \leq R_1 \leq R_{\text{cr},1}, 0 \leq R_2 \leq R_{\text{cr},2}\}$  and  $R_d^*$  be the largest sum rate at which the optimal solution  $(R_1, R_2)_{\text{opt}}$  of the problem  $\mathcal{P}_1$  belongs to the subregion  $\mathcal{R}_1$ . When the rate is less than the critical rate, the optimal value of  $\rho$  is equal to 1 and the RCEE for the link  $L_{k \in \mathcal{T}}$  of the TWRC can be written as

$$\tilde{E}_{r,k}(R_k) = \tilde{E}_{0,k}(1) - 2R_k = 2(R_{0,k} - R_k). \quad (47)$$

Therefore, for the sum rate  $R \leq R_d^*$ , the problem  $\mathcal{P}_1$  can be rewritten as

$$\mathcal{P}_1 = \left\{ \begin{array}{ll} \max_{R_1, R_2} & \min_{k \in \mathcal{T}} (R_{0,k} - R_k) \\ \text{s.t.} & R_1 + R_2 = R \\ & 0 \leq R_1 \leq R_{\text{cr},1}, 0 \leq R_2 \leq R_{\text{cr},2} \end{array} \right. \quad (48)$$

which is equivalent to

$$\mathcal{P}_1 = \left\{ \begin{array}{ll} \max_{R_1} & \min \{R_{0,1} - R_1, R_{0,2} - R + R_1\} \\ \text{s.t.} & 0 \leq R_1 \leq R \leq R_{\text{cr},1} + R_{\text{cr},2}. \end{array} \right. \quad (49)$$

Without loss of generality, we assume  $R_{0,2} \geq R_{0,1}$ , and we can consider two different cases as follows:

- When  $R \leq R_{0,2} - R_{0,1}$ , we have  $R_{0,2} - R + R_1 \geq R_{0,1} - R_1$  and

$$\min \{R_{0,1} - R_1, R_{0,2} - R + R_1\} = R_{0,1} - R_1 \leq R_{0,1}. \quad (50)$$

Thus, in this case, the optimal rate pair is

$$(R_1, R_2)_{\text{opt}} = (0, R). \quad (51)$$

- When  $R \geq R_{0,2} - R_{0,1}$ , we need to consider two additional cases.

If  $R_{0,1} - R_1 \geq R_{0,2} - R + R_1$  or  $R_1 \leq \frac{1}{2}(R - R_{0,2} + R_{0,1})$ , then

$$\begin{aligned} & \min \{R_{0,1} - R_1, R_{0,2} - R + R_1\} \\ &= R_{0,2} - R + R_1 \\ &\leq R_{0,2} - R + \frac{R - R_{0,2} + R_{0,1}}{2} = \frac{-R + R_{0,2} + R_{0,1}}{2}. \end{aligned} \quad (52)$$

Therefore, the optimal rate pair is given by

$$(R_1, R_2)_{\text{opt}} = \left( \frac{R + R_{0,1} - R_{0,2}}{2}, \frac{R - R_{0,1} + R_{0,2}}{2} \right). \quad (53)$$

If  $R_{0,1} - R_1 \leq R_{0,2} - R + R_1$  or  $R_1 \geq \frac{1}{2}(R - R_{0,2} + R_{0,1})$ , then

$$\begin{aligned} & \min \{R_{0,1} - R_1, R_{0,2} - R + R_1\} \\ &= R_{0,1} - R_1 \\ &\leq R_{0,1} - \frac{R - R_{0,2} + R_{0,1}}{2} = \frac{-R + R_{0,2} + R_{0,1}}{2}. \end{aligned} \quad (54)$$

Therefore, the optimal rate pair is given by

$$(R_1, R_2)_{\text{opt}} = \left( \frac{R + R_{0,1} - R_{0,2}}{2}, \frac{R - R_{0,1} + R_{0,2}}{2} \right). \quad (55)$$

Since  $(R_1, R_2)_{\text{opt}}$  should belong to  $\mathcal{R}_1$ , we can find the decisive sum rate  $R_d^*$  as (28) from the fact that

$$\left\{ \begin{array}{l} \frac{R + R_{0,1} - R_{0,2}}{2} \leq R_{\text{cr},1} \\ \frac{R - R_{0,1} + R_{0,2}}{2} \leq R_{\text{cr},2}. \end{array} \right. \quad (56)$$

From (51), (53), and (55), we arrive at the desired result (27).

### D. Proof of Theorem 3

For any  $t \in \mathbb{R}_+$ , the upper-level set of  $E_{r,k}^{\text{int}}(R_k)$  that belongs to  $\mathcal{S}$  is given by

$$\begin{aligned} & U \left( E_{r,k}^{\text{int}}, t \right) \\ &= \left\{ \boldsymbol{\psi} \in \mathbb{R}_+^2 : \frac{\psi_k^2}{\psi_k^2 \alpha_k + (p_R + \psi_1^2 \psi_2^2 / \psi_k^2) \alpha_1 \alpha_2 / \alpha_k + 1} \geq v_k \right\} \\ &= \left\{ \boldsymbol{\psi} \in \mathbb{R}_+^2 : \frac{\boldsymbol{\psi}^T \mathbf{e}_k}{\sqrt{v_k}} \geq \sqrt{1 + p_R \alpha_1 \alpha_2 / \alpha_k + \|\mathbf{A}\boldsymbol{\psi}\|^2} \right\} \\ &= \left\{ \boldsymbol{\psi} \in \mathbb{R}_+^2 : \left[ \begin{array}{c} \boldsymbol{\psi}^T \mathbf{e}_k / \sqrt{v_k} \\ \mathbf{A}\boldsymbol{\psi} \end{array} \right] \succeq_{\kappa} \mathbf{0} \right\} \end{aligned} \quad (57)$$

with

$$\mathbf{A} \triangleq \text{diag}(\sqrt{\alpha_1}, \sqrt{\alpha_2}) \quad (58)$$

$$v_k \triangleq \frac{(1+\rho)}{p_R \alpha_1 \alpha_2} \left[ \exp\left(\frac{t + 2\rho R_k}{\rho}\right) - 1 \right]. \quad (59)$$

It is clear that  $U\left(E_{r,k}^{\text{int}}, t\right)$  is a convex set since it can be represented as an SOC. Since the upper-level set  $U\left(E_{r,k}^{\text{int}}, t\right)$  is convex for every  $t \in \mathbb{R}_+$ ,  $E_{r,k}^{\text{int}}(\mathbf{p}, \rho, R_k)$  is, thus, quasi-concave.<sup>13</sup>

We now show that  $E_{r,k}^{\text{int}}(\mathbf{p}, \rho, R_k)$  is not concave by contradiction. Since the function  $\ln(\cdot)$  is a monotonic function, we simply need to show that

$$f_k(\boldsymbol{\psi}) = \frac{\psi_k^2}{\psi_k^2 \alpha_k + (p_R + \psi_1^2 \psi_2^2 / \psi_k^2) \alpha_1 \alpha_2 / \alpha_k + 1}$$

is not concave. We consider  $\boldsymbol{\psi}_a$  and  $\boldsymbol{\psi}_b$  such that  $\boldsymbol{\psi}_a = \zeta \mathbf{e}_k$  and  $\boldsymbol{\psi}_b = \delta \zeta \mathbf{e}_k$  for  $0 \leq \zeta \leq \sqrt{P}$  and  $0 < \delta < 1$ . Clearly,  $\boldsymbol{\psi}_a$  and  $\boldsymbol{\psi}_b$  are feasible solutions of  $\mathcal{P}_2$ . For any  $\lambda \in [0, 1]$ , we have

$$f_k(\lambda \boldsymbol{\psi}_a + (1 - \lambda) \boldsymbol{\psi}_b) = \left( \alpha_k + \frac{1 + p_R \alpha_1 \alpha_2 / \alpha_k}{\zeta^2 [\lambda + \delta(1 - \lambda)]^2} \right)^{-1} \triangleq g_k(\zeta) \quad (60)$$

where  $g_k(\zeta)$  is clearly convex in  $\zeta$ . Due to convexity of  $g_k(\zeta)$ , the following inequality must hold

$$g_k(\lambda \zeta_a + (1 - \lambda) \zeta_b) \leq \lambda g_k(\zeta_a) + (1 - \lambda) g_k(\zeta_b). \quad (61)$$

Now, by letting  $\zeta_a = \zeta / (\lambda + \delta(1 - \lambda))$  and  $\zeta_b = \delta \zeta / (\lambda + \delta(1 - \lambda))$ , we can rewrite (61) as

$$f_k(\lambda \boldsymbol{\psi}_a + (1 - \lambda) \boldsymbol{\psi}_b) \leq \lambda f_k(\boldsymbol{\psi}_a) + (1 - \lambda) f_k(\boldsymbol{\psi}_b). \quad (62)$$

Thus, we have showed that there exist  $\boldsymbol{\psi}_a, \boldsymbol{\psi}_b \in \mathbb{R}_+^2$  and  $\lambda \in [0, 1]$  such that (62) holds. By contradiction,  $f_k(\boldsymbol{\psi})$  is not a concave function on  $\mathbb{R}_+^2$ . Therefore, it follows that  $E_{r,k}^{\text{int}}(\mathbf{p}, \rho, R_k)$  is also not concave.

Since the nonnegative weighted minimum of quasi-concave functions is quasi-concave [22],  $E_r^{\text{int}}(\mathbf{p}, \rho, R_1, R_2)$  is also quasi-concave. Furthermore,  $\mathcal{P}_2$  is a quasi-concave optimization problem since the constraint set in  $\mathcal{P}_2$  is convex in  $\boldsymbol{\psi}$ .

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<sup>13</sup>Note that a concave function is also quasi-concave.

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